UNIT-III

Logic and Proofs : Proposition, Conjunction, Disjunction, Negation, Compound proposition, Conditional propositions (Hypothesis, conclusion, necessary and sufficient condition) and Logical equivalence, De Morgan's law, Tautology and contradiction, quantifiers, universally quantified statements, component of a Mathematical system (axiom, definitions, undefined terms, theorem, lema and corollary), proofs (direct proofs, indirect proofs, proof by contra-positive), Mathematical Induction.

Boolean Algebra: Definition and Laws of Boolean Algebra, Bookan functions, Simplification of Boolean functions. Special forms of Boolean functions, Application of Boolean algebra(open and c'osed switches, switches in series and parallel). Logic gates and Circuits.

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LOGIC AND PROOFS

INTRODUCTION

Logic is concerned with the method of reasoning. Logical reasoning is used in many field like to prove theorem in mathematics, to programming in computer science, many logical expression are used in algorithm. Prepositional logic is the area of logic that deals with prepositional. A truth table displays the relationships between the truth values of propositions.

PROPOSITION CACULUS

Propositional calculus is the branch of mathematics that is used to describe a logical system. Using logic a person can solve specified problem. The method of drawing conclusions in some manner is called logic. Logic has provided us a systematic way to think over a specified problem.

The science of art of exact reasoning or art of pure and formal thought according to which the processes of pure thinking should be conducted. Logic provides rules and techniques and thoughts for determining whether a given argument is valid. A logical system consists of:

- Statement: It is either a meaningful declarative sentence that is either true or false

- Truth table: Truth table shows the truth results of logical connectives.

- Definition: A definition is a statement that explains the meaning of a term.

PROPOSITION

A proposition is a meaningful statement that is either true or false, but not both. The value 'true' and 'false' is denoted by T and F respectively.

There are two types of propositions:

- **Simple:** The statement without connectives whose truthness depends on a single statement is called simple statements.

Eg. He is a doctor.

- **Compound:** The statement formed by joining the simple statements and whose truthness depends on combination of two or more statements is called compound statements.

Eg. Delhi is capital of India and capital of Rajasthan is Jaipur.

LOGICAL CONNECTIVE

Connectives are the symbols or words that are used for making compound statements.

Here we shall discuss about the some simple and basic connectives (conjunction, disjunction and negation), conditional connectives (if then) and bi conditional connectives (iff.).

Type	Connectives	Symbols	Use
Simple	And (Conjunction)	$\wedge$	$p \wedge q$
Simple	Or (Disjunction)	$\vee$	$p \vee q$
Simple	Negation	$\sim$	$\sim p$
Conditional	If then	$\rightarrow$	$p \rightarrow q$
Bi-conditional	If and only if	$\leftrightarrow$	$p \leftrightarrow q$

Conjunction ($p \wedge q$, p and q): Let p and q be two simple statements, then conjunction of p and q is the compound statements 'p and q'.

Conjunction of p and q is denoted by $p \wedge q$.

Truth table shows that $p \wedge q$ is true when both p and q are true otherwise it will be false.

Eg. P: He is a doctor

q: His wife is an engineer.

$p \wedge q$: He is a doctor and his wife is an engineer.

Truth table for conjunction:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction ($p \vee q$, p or q): Let p and q be two simple statements, then disjunction of p and q is the compound statements 'p or q'.

Conjunction of p and q is denoted by $p \vee q$.

Truth table shows that $p \vee q$ is true when one of the p and q are true or both p and q are true otherwise it will be false if both are false.

Eg. P: He is a doctor

q: His wife is an engineer.

$p \vee q$: He is a doctor or his wife is an engineer.

Truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation ($\sim p$, not p): Let p be a simple statement, then negation of p is 'not p'.

Negation of p is denoted by $\sim p$.

Truth table shows that $\sim p$ is true when p is false and p is false if $\sim p$ is true.

Eg. P: He is a doctor.

$\sim q$: He is not a doctor.

Truth table for negation:

p	$\sim q$
T	F
F	T

Implication ($p \rightarrow q$, if p then q): Let p and q be two simple statements, then the statement "if p then q" is called conditional statement or implication.

Implication of p and q is denoted by $p \rightarrow q$.

The statement p is called antecedent or hypothesis and the statement q is called consequent or conclusion.

Logically $p \rightarrow q \equiv \sim p \vee q$.

Truth table shows that $p \rightarrow q$ is false only when p is true and q is false otherwise it will be true.

Eg. p: He is thirsty

q: He will drink water.

$p \to q$: If he is thirsty then he will drink water.

Truth table for implication:

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Important Result related to implication $p \to q$:

- Propositions that are related to an implication $p \to q$
- Converse of an implication $p \to q$ is $(q \to p)$
- Contra positive of an implication $p \to q$ is $(\sim q \to \sim p)$
- Inverse of an implication $p \to q$ is $(\sim p \to \sim q)$

Eg. 1 Consider the following sentence

P: He is thirsty.

q: He will drink water.

Find the implication, its converse, contra positive and inverse of the given sentences.

Sol: Implication $(p \to q)$ = If he is thirsty then he will drink water.

Converse $(q \to p)$ = If he drink water then he is thirsty.

Contra positive $(\sim q \to \sim p)$ = If he don't drink water then he is not thirsty.

Inverse $(\sim p \to \sim q)$ = If he is not thirsty then he will not drink water.

Bi-conditional ($p \leftrightarrow q$, p if and only if q): Let p and q be two simple statements, then the statement "p if and only q" is called Bi-conditional statement.

Implication of p and q is denoted by $p \leftrightarrow q$.

Logically $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$.

Truth table shows that $p \leftrightarrow q$ is true only when both p and q are true or when both p and q are false.

Eg. p: He will go to market.

q: He will have complete his work.

$p \leftrightarrow q$: He will go to market if and only if he will have complete his work.

Truth table for Bi-conditional:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Note: Iff is abbreviation of **If** and only **if**. So these have same meaning and used for same sense.

PROPOSITIONAL EQUIVALENCES

Compound propositions that always have the same truth value are called logically equivalent (denoted by $\equiv$).

implications

Logical equivalences involving implications

Logical Equivalences Involving Implications
$p \to q \equiv \sim p \vee q$
$p \to q \equiv \sim q \to \sim p$
$p \vee q \equiv \sim p \to q$
$p \wedge q \equiv \sim (p \to \sim q)$
$\sim (p \to q) \equiv p \wedge \sim q$
$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$
$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$
$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$
$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

Logically equivalences involving bi-conditional

Logical Equivalences Involving Bi-conditionals
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$
$p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

Eg. 2. Test the Truthiness of following statement with the help of Truth Table.

(a) (b) (c) R (d)

Sol:

(a) Let's consider $p \rightarrow (q \wedge r) \equiv A$ &

$(p \rightarrow q) \wedge (p \rightarrow r) = B$

Truth Table

p	q	r	$q \wedge r$	$p \rightarrow q$	$p \rightarrow r$	A	B
T	T	T	T	T	T	T	T
T	T	F	F	T	F	F	F
T	F	T	F	F	T	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

So that A = B

Hence $p \rightarrow (q \wedge r) \equiv (p \rightarrow q) \wedge (p \rightarrow r)$

(b) $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$

Sol: Let's consider

$p \rightarrow (q \vee r) = A$ &

$(p \rightarrow q) \vee (p \rightarrow r) = B$

p	q	r	$q \vee r$	$p \rightarrow q$	$p \rightarrow r$	A	B
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

So that A = B

$\Rightarrow p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$

(c) $(p \rightarrow q) \vee r \equiv (p \vee r) \rightarrow (q \vee r)$

Sol:

$(p \rightarrow q) \vee r = A$

$(p \vee r) \rightarrow (q \vee r) = B$ [Let's consider]

Truth Table:

p	q	r	$p \vee r$	$q \vee r$	$p \rightarrow q$	p	q
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	T	T	F	T	T
T	F	F	T	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	F	T	T	T

(d) $(p \rightarrow q) \wedge (q \rightarrow p) \equiv p \leftrightarrow q$

Sol: Let's consider $(p \rightarrow q) \wedge (q \rightarrow p) = A$

$p \leftrightarrow q = B$

Truth Table:

p	q	$p \rightarrow q$	$q \rightarrow p$	A	B
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

Algebra of propositions

(1) Double Negation law

Proof: Truth Table $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

(2) Idempotent Laws:

(a) $p \vee p = p$

(b) $p \wedge p = p$

Proof: Truth-Table

p	$p \wedge q$	$p \vee q$
T	T	T
F	F	F

(3) Associative laws

(a) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \equiv (p \wedge q) \wedge r$

Proof: Truth Table

(a)

p	q	r	$(p \vee q)$	$(p \vee q) \vee r$	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T	T	T
T	T	F	T	T	T	T
T	F	T	T	T	T	T
T	F	F	T	T	F	T
F	T	T	T	T	T	T
F	T	F	T	T	T	T
F	F	T	F	T	T	T
F	F	F	F	F	F	F

(b)

p	q	r	$(p \wedge q)$	$(p \wedge q) \wedge r$	$q \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	F	T	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

(4) Commutative laws

(a) $p \vee q \equiv q \vee p$

(b) $p \wedge q \equiv q \wedge p$

Proof:

(a)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

(b)

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(5) Distributive laws:

(a) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

(b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Sol: Let's consider

$p \vee (q \wedge r) = A$

& $(p \vee q) \wedge (p \vee r) = B$

(a)

p	q	r	$(p \vee q)$	$(p \vee r)$	$(q \vee r)$	A	B
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

(b) Let's consider $p \wedge (q \vee r) = A$

$(p \wedge q) \vee (p \wedge r) = B$

Truth-Table:

p	q	r	$(p \wedge q)$	$(p \wedge r)$	$q \wedge r$	P	Q
T	T	T	T	T	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	F	F	F
F	T	T	F	F	T	F	F
F	T	F	F	F	T	F	F
F	F	T	F	F	T	F	F
F	F	F	F	F	F	F	F

(6) De Morgan's law

(a) $\sim(p \vee q) \cong \sim p \wedge \sim q$

(b) $\sim(p \wedge q) \cong \sim p \vee \sim q$

Proof:

(a)

p	q	$(p \vee q)$	$\sim(p \vee q)$	$\sim p$	$\sim q$	$\sim p \wedge \sim q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

(b)

p	q	$(p \wedge q)$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Tautologies: A compound proposition that is always true is known as tautology.

Eg. $p \vee \sim p$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

$\wedge$ = T

Eg.3 Prove that $(p \wedge q) \rightarrow p$ is a Tautology.

Sol: Truth-Table

p	q	$(p \wedge q)$	$(p \wedge q) \rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

So, $(p \wedge q) \rightarrow p$ is a Tautology.

Contradiction: A compound proposition that is always false is known as contradiction.

Eg. $p \wedge \sim p$

p	$\sim p$	$P \wedge \sim p$
T	F	F
F	T	F

Eg.4 Prove that $\sim[(p \wedge q) \rightarrow p]$ is a Contradiction.

Sol: Truth-Table

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$	$\sim[(p \wedge q) \rightarrow p]$
T	T	T	T	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

So, $\sim[(p \wedge q) \rightarrow p]$ is a Contradiction.

PREDICATES:

A predicate is a word or words in a sentence which express the property or nature of the subject.

A propositional function $p(x)$ has two parts: the variable x is the subject of the statement, and the predicate p is the property that the subject can have.

Eg. $p(x) = x$ is divisible by 2.

Here 'is divisible by 2' is predicate and x is variable.

The truth value of $p(x)$ depends on the value of x.

It can be determined when x is assigned a value.

QUANTIFIERS:

Quantifiers can be used to indicate how frequently a predicate $p(x)$ is true. In a propositional function the truthness of the $p(x)$ depends on the values of the variables. When these variables are assigned the values then we can determine that resulting statement is true or false. There are two possibilities that

- For all values, the statement is always true or
- There may be some values for which the statement is false.

On this basis Quantifiers can be categorized into two parts:

Universal Quantifier [$\forall x\, p(x)$]: In a propositional function if for every value of variable in a particular range, predicate is always true then such a statement can be expressed using the universal quantifier.

Let p (x) be a predicate defined in a set A such that

$\forall (x \in A)\, p(x)$

Which is "for every x in A, p (x) is true statement" is called universal quantifiers.

Eg.5 The proposition $\forall (n \in N)\ (n+5>4)$ is true since for every natural number the predicate $p(x)$ is always true.

Existential Quantifier [$\exists x\, p(x)$]:In a propositional function if there exist atleast one value for which predicate is true then such a statement can be expressed using the existential quantifier.

Let $p(x)$ be a predicate (defined in a set A) such that

$\exists (x \in A)\, p(x)$

Which is "There exist x in set A such that p (x) is true statement" is called a existential quantifier.

Eg. The proposition $\exists (n \in N)\ (n+1<4)$ is true since for natural number 1 R 2 the predicate p (x) is true.

In briefly we can understand the quantifier the using the following table.

Statement	When True?	When False?
$\forall x\, p(x)$	$P(x)$ is true for every x.	There is an x for which $p(x)$ is false
$\exists x\, p(x)$	There is at least one x for which $p(x)$ is true.	$P(x)$ is false for every x.

Eg.6 Given following statements

p (x): x>0

g (x): x is odd

r (x): x is perfect square

s (x): x is divisible by 2

Write following statements in symbolic form

a) At least one integer is odd.

b) There exists a positive integer that is odd.

c) If is odd, then x is not divisible by 2.

d) No odd integer is divisible by 2.

Sol:

a) $\exists (x \in I)\, g(x)$

b) $\exists x\, [p(x) \wedge g(x)]$

c) $\forall x [g(x) \rightarrow (\sim s(x))]$

d) $\forall x [g(x) \rightarrow (\sim s(x))]$

Components of Mathematical System

The following are the components of mathematical system

1. **Undefined Terms:** Any term used in describing the whole is said to be undefined term.

   Eg. Points in geometry.

2. **Axioms:** The proposition that is accepted without any proof is known as axioms.

   Eg. $A+B = B+A$

3. **Definition:** Definition is a symbol or set of word which converge some expression that is too lengthy.

   Eg. The definition of square the quadrilateral whose all side are equal and each angle is of .

4. **Theorem;** A theorem is a general conclusion which can be derived by axioms, definition, undefined term and another previously proved theorem.

   Eg. The sum of angle of a triangle is 180°

5. **Lemma:** The small theorem which is used to prove another theorem is known as lemma.

6. **Corollary:** A corollary is quickly proved theorem from some other theorem.

7. **Proof:** A proof is a logical argument that establishes that a specific statement, proposition, or mathematical formula is true.

It consists of a set of assumptions (also called premises) that are combined according to logical rules in order to establish a valid conclusion. Three types of proof are used for proving theorem.

(a) Direct Proof: In a direct proof, a given conclusion can be shown to be true. The conclusion is established by logically combining the axioms, definitions, and earlier theorems.

In other words we can say that in direct proof we always consider the conclusion is true and for proving it true we uses axioms, other proofs, lemma and facts.

For example by direct proof we can prove that"sum of angle of triangle is 180° ". Now for proving it we uses others theorem, lemma and other facts and finally prove it.

Eg.7 Let the following statements be true.

If I am weak, then I will study.

I will play or I will not study.

I will not play.

Show that the statement 'I am not weak' is true.

Sol: Let p, q and r represent the following statements:

P: I am weak.

q: I will play.

r: I will study.

The given statement can be written in form of p, q, r, as

$p \to q$, $q \vee \sim r$ and $\sim r$.

According to question we want to prove that $\sim p$ is true. So using direct proof first we know that $\sim r$ is true. But between q and $\sim r$ one should be true so q is false or $\sim q$ is true now $p \to q$ shows that $\sim q \to \sim p$ is true so $\sim p$ is true.

(a) Indirect Proof: In an indirect proof, a given conclusion can be also not to be false and, therefore, it will be true.

In other word we can say that first we consider the conclusion is false and then proving this assumption adding with the statements of other premises of the theorem which leads a contradiction. These contradictions show that our assumption is wrong and conclusion is true. For example there are some theorems which can't be proved using direct proof just as" $\sqrt{2}$ is irrational number". Now for proving it we assume that $\sqrt{2}$ is rational number and prove this assumption and finally obtain a contraction which shows that our assumption is wrong. So $\sqrt{2}$ is irrational number.

Eg.8 Let the following statements be true.

If I am lazy, then I do not study.

I study or I enjoy myself.

I do not enjoy myself.

Show that the statement 'I am not lazy' is true. **[MCA 2008, MW: 1]**

Sol: Let p, q, and r represent the following statements:

p: I am lazy.

q: I study

r: I enjoy myself.

The given statement can be written in form of p, q, r as

$p \to \sim q$, $q \vee r$ and $\sim r$.

According to question we want to prove that $\sim p$ is true. So using indirect proof assumes that p is true. Since p is true then using $p \to \sim q$, $\sim q$ is also true i.e. q is false. Now r is also false but either q or r is true. This is not possible. So it's a contradiction. Our assumption is wrong. So $\sim p$ is true.

(c) **Proof by contra positive:** If an implication statement is proved by its contra positive, it is known as proof by contra positive. i.e. let's consider a sentence is $p \to q$ then its contra positive will be $\sim q \to \sim p$. so $p \to q$ can be proved by $\sim q \to$ np is true.

Eg. Prove that "if 3n+2 is odd, then n is odd" using the proof by contra positive.

Sol: Assume that

p: 3n+2 is odd.

q: n is odd.

Now for proving $p \to q$ by $\sim q \to \sim p$ is true. So $\leftarrow . \sim q$ is nis not odd i.e. n is even, then n = 2k for some integer k. now

3n+2

= 3(2k) +2

= 2(3k+1), which is also even, so it is not odd. So $\sim q \to \sim p$ is true. So we can say that $p \to q$

ARGUMENTS

An argument is an assertion that a finite number of statements $p_1, p_2 \ldots p_n$ called premises or hypotheses gives another statement denoted by q is called the conclusion.

$$p_1 \cdot p_2 \cdots p_n \to q$$

Validity of an argument depends on all the premises i.e. if all the premises are true then conclusion is also true. In other words we can say that if $p_1 \cdot p_2 \cdots p_n \to q$ shows the tautology then argument is valid. There are following methods to check the validity of arguments:

Truth table method: In this method validity of argument is checked by making the truth table for the given arguments.

Simplification method: Using the standard method of simplification we reduce the given

argument to T.

Inference rule method: In this method we simplify the given argument using some rules known as rules of inference, these rules are:

Rule of Inference	Tautology	Name
$\underline{p}$ $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$\underline{p \wedge q}$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p $\underline{q}$ $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
P $\underline{p \to q}$ $\therefore q$	$[p \wedge (p \to q)] \to q$	Modus ponens
$\sim q$ $\underline{p \to q}$ $\therefore \sim p$	$[\sim q \wedge (p \to q)] \to \sim p$	Modus tollens
$p \to q$ $\underline{q \to r}$ $\therefore p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$	Hypothetical Syllogism
$p \vee q$ $\underline{\sim p}$ $\therefore q$	$[(p \vee q) \wedge \sim p] \to q$	Disjunctive Syllogism
$p \vee q$ $\underline{\sim p \vee r}$ $\therefore q \vee r$	$[(p \vee q) \wedge (\sim p \vee r)] \to (q \vee r)$	Resolution

Questions

Very Short Type Questions:

1. Define implication.
2. What is proposition?
3. Define logic.
4. What is quantifier?

5. Define predicates.
6. What is axiom?
7. Define theorem.
8. What do you know about proof?

Short Type Questions:

1. Define proposition and its type.
2. What are connectives?
3. Define the type of proof.
4. Define tautology and contra dictions with example.
5. Define the term predicates and propositional function.
6. Explain logical equivalence of two propositions by using suitable example.
7. Prove the Demorgon's law using truth table.

Long Type Questions:

1. What is argument? How can we check the validity of arguments?
2. Write short notes on proof and indirect proof and proof by contra positive with example.
3. Write short notes on Propositional calculus.
4. Prove that:

(a) $(p \vee q) \Rightarrow r \equiv (p \Rightarrow r) \wedge (q \Rightarrow r)$.

(b) $p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$.

(c) $(p \to q) \wedge (p \to r) \equiv p \to (p \wedge r)$.

5. Show that the propositions $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.
6. Define the Quantifiers and its types.

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7

BOOLEAN ALGEBRA

Introduction

George Boole developed the concept of Boolean Algebra. It is used for developing mathematical logic. It is used for design of electronic computers, switching circuits etc. In Boolean Algebra, logical reasoning is defined by symbols. It is a algebra for manipulation of objects that can take only two values as true and false which is as similar as computer are build as collections of switches that are either on or off.

Definition

A non empty (or non void) set B together with two binary operation addition (+) and multiplication (-) and unary operations complementation (.) on B is said to be Boolean Algebra. The Boolean Algebra is based on certain laws as follows :

(a) Commutative laws

$a.b = b.a$

$a + b = b + a$

(b) Distributive laws

$a.(b + c) = a.b + a.c$

$a + (b.c) = (a + b).(a + c)$

(c) Identity laws

$a.1 = a$

$a + 0 = a$

(d) Complementation laws

$a+a'=1$

$a.a'=0$

(e) Idempotent laws

$a+a=a$

$a.a=a$

(f) Boundedness laws

$a+1=1$

$a.0=0$

(g) Absorption laws

$a+ab=a$

$a(a+b)=a$

(h) Demorgan's laws

$(a+b)'=a'b'$

$(ab)'=a'+b'$

Eg. 1 Prove absorption laws

(i) $a+ab=a$ **and**

(ii) $a(a+b)=a$

Proof: (i) L.H.S. $a+ab$

$=a(1+b)$

$=a.(1) \qquad \because 1+b=1$

$=a \qquad \because a.1=a$

(ii) L.H.S. $a(a+b)$

$=a.a+ab$

$=a+ab \qquad \because a.a=a$

$=a(1+b)$

$=a.1 \qquad \because (1+b)=1$

$=a$

Eg. 2 Prove the following

(i) $(a+b).(b+c).(c+a)=ab+bc+ca$

(ii) $a+a'b=a+b$

(iii) $a.(ab)=ab$

(iv) $a+(a+b)=a+b$

Sol. **(i)** L.H.S. $(a+b).(b+c).(c+a)$

$=(a+b).[(c+b)(c+a)]$ (Using Commutative Law)

$=(a+b).[c+ba]$

$=(a+b).c+(a+b).ba$

$=ac+bc+aab+abb$

$=ac+bc+ab+ab \qquad (\because aa=a)$

$=ac+bc+ab \qquad (\because a+a=a)$

= R.H.S.

(ii) L.H.S. $a+a'b$

$=(a+a')(a+b)$ (Using Distributive law)

$=1(a+b)$ (Using Complementation law)

$=a+b$

= R.H.S.

(iii) L.H.S. $a(a.b)$

$= a.a.b$

$= (a.a)b$

$= ab =$ R.H.S.

(iv) L.H.S. $a+(a+b)$

$= (a+a)+b$

$= a+b \qquad (\because a+a=a)$

Eg 3. **Prove that**

(i) $(a+b).(a'+c) = a.c + a'.b$

(ii) $(a+b).(b'+c)+b.(b'+c') = a.b'+a.c+b$

(iii) $abc+a'+b'+c'=1$

(iv) $abc+abc'+ab'c+a'bc = ab+bc+ca$

(v) $(a+b)'+(a+b')' = a'$

Sol. (i) L.H.S. $(a+b).(a'+c)$

$= a.a'+a'b+ac+bc$

$= 0+a'b+ac+bc \qquad (\because aa'=0)$

$= a'b+ac+1.bc$

$= a'b+ac+(a+a').bc$

$= a'b+ac+abc+a'bc$

$= a'b+a'bc+ac+abc$

$= a'b(1+c)+ac(1+b) \qquad (\because 1+a=1)$

$= a'b+ac$

= R.H.S.

(ii) L.H.S. $(a+b).(b'+c)+b.(b'+c')$

$= a.b'+b.b'+a.c+b.c+b.b'+b.c'$

$= a.b'+0+a.c+b.c+0+b.c'$

$= a.b'+a.c+b.c+b.c' \qquad (\because aa'=0)$

$= a.b'+a.c+b.(c+c') \qquad (\because a+a'=1)$

$= a.b'+a.c+b$

(iii) L.H.S. $abc+a'+b'+c'$

$= abc+(a.b)'+c'$ (Using Demorgan's Law)

$= abc+(a.b.c)'$ (Using Demorgan's Law)

$= 1 \qquad (\because a+a'=1)$

= R.H.S.

(iv) L.H.S. $abc+abc'+ab'c+a'bc$

$= ab(c+c')+ab'c+a'bc$

$= ab+ab'c+a'bc \qquad (\because a+a'=1)$

$= a(b+b'c)+a'bc$

$= a[(b+b').(b+c)]+a'bc \qquad (\because a+a'=1)$

$= a(b+c)+a'bc$

$= ab+ac+a'bc$

(iii) L.H.S. $a(a.b)$

$= a.a.b$

$= (a.a)b$

$= ab =$ R.H.S.

(iv) L.H.S. $a+(a+b)$

$= (a+a)+b$

$= a+b \qquad (\because a+a=a)$

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(v) $(a+b)'+(a+b')' = a'$

Sol. (i) L.H.S. $(a+b).(a'+c)$

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$= 0+a'b+ac+bc \qquad (\because aa'=0)$

$= a'b+ac+1.bc$

$= a'b+ac+(a+a')bc$

$= a'b+ac+abc+a'bc$

$= a'b+a'bc+ac+abc$

$= a'b(1+c)+ac(1+b) \qquad (\because 1+a=1)$

$= a'b+ac$

= R.H.S.

(ii) L.H.S. $(a+b).(b'+c)+b.(b'+c')$

$= a.b'+b.b'+a.c+b.c+b.b'+b.c'$

$= a.b'+0+a.c+b.c+0+b.c'$

$= a.b'+a.c+b.c+b.c' \qquad (\because aa'=0)$

$= a.b'+a.c+b.(c+c') \qquad (\because a+a'=1)$

$= a.b'+a.c+b$

(iii) L.H.S. $abc+a'+b'+c'$

$= abc+(a.b)'+c'$ (Using Demorgan's Law)

$= abc+(a.b.c)'$ (Using Demorgan's Law)

$= 1 \qquad (\because a+a'=1)$

= R.H.S.

(iv) L.H.S. $abc+abc'+ab'c+a'bc$

$= ab(c+c')+ab'c+a'bc$

$= ab+ab'c+a'bc \qquad (\because a+a'=1)$

$= a(b+b'c)+a'bc$

$= a[(b+b').(b+c)]+a'bc \qquad (\because a+a'=1)$

$= a(b+c)+a'bc$

$= ab+ac+a'bc$

$= ab + c[a + a'b]$

$= ab + c[(a + a').(a + b)]$

$= ab + c(a + b)$

$= ab + ac + cb$

= R.H.S.

(v) L.H.S. $(a+b)'+(a+b')'$

$= a'.b'+a'.(b')'$ (Using Demorgan's Law)

$= a'.b'+a'.b$

$= a'.(b'+b)$ $(\because a+a'=1)$

$= a'$

Boolean Functions and Boolean Expressions

Boolean Expression:

Any expression built up with a finite set of variable or its complement by applying operations addition (+) or multiplication (.).

Boolean Functions:

A Boolean expression $f(x_1, x_2 \ldots x_n)$ of n variables is a function from B^n to B if it can be determined by the Boolean expression $\alpha(x_1, x_2 \ldots x_n)$

Eg. $f(x, y) = x' + y$ then

$f(0,1) = 0'+1$

$= 1+1 = 1$

$f(1,0) = 1'+0$

$= 0+0 = 0$

Different Boolean expression may determine the same boolean functions. In these case the Boolean expression will be equivalent standard (special form of Boolean functions).

Minterm:

In a Boolean expression of n variable, the product of all n variables (either in complemented or uncomplemented form) are known as minterm.

For eg. If we have two variables a and b then possible minterms are $ab, a'b, ab'$ and $a'b'$. The number of minterms are 2^n.

Maxterm:

In a Boolean expression of n variable, the sum of all n variables (either in complemented or uncomplemented form) are known as maxterm.

For eg. If we have two variables a and b then possible maxterms are $a+b, a'+b, a+b'$ and $a'+b'$. The number of maxterms are 2^n.

There are basically two special forms of Boolean functions.

1. DNF (Disjunctive normal form) (sum of products)
2. CNF (Conjuctive normal form) (product of sum)

1. Disjunctive normal form (DNF) : This is also known as sum of products since every product term are added in this form.

For eg. $f(x, y, z) = x + yz + x'yz$ is a DNF.

Eg. Express $(x+y+z)(xy+x'z)'$ in DNF

Sol. We have $(x+y+z)(xy+x'z)'$

$= (x+y+z)((xy)'+(x'z)')$ (Using Demorgan's Law)

$= (x+y+z)(x'+y')(x+z')$ (Using Demorgan's Law)

$= (x+y+z)(xx'+x'z'+xy'+y'z')$

$= (x+y+z)(0+x'z'+xy'+y'z') \qquad (\because aa'=0)$

$= xx'z'+xxy'+xy'z'+x'yz'$

$\quad +xyy'+yy'z'+x'zz'+xy'z+y'zz'$

$= 0+xy'+xy'z'+x'yz'+0+0+0+xy'z+0$

$(\because aa'=0 \,\&\, aa=a)$

$= xy'(1+z')+xy'z'+x'yz'$

$= xy'+xy'z'+x'yz'$

$= xy'(1+z')+x'yz'$

$= xy'+x'yz' \qquad (\because 1+a=1)$

Principle Disjunctive Normal Form

If in a DNF every term contain every variable (either in complemented or uncomplemented form). This is known as Principle Disjunctive Normal Form or standard sum of product (SSOP). Every term is known as minterm.

Conversion of DNF into PDNF:

To convert DNF into PDNF first we find the term with missing variable and multiply the addition of missing variable and its complement, the simply and drop the common term.

Eg. **Convert** $f(x,y,z)=xy'+x'yz'$ **into PDNF**

Sol. $f(x,y,z)=xy'+x'yz'$

$= xy'(z+z')+x'yz'$

$= xy'z+xy'z'+x'yz'$

2. Conjuctive Normal Form CNF (product of sum)

This is also known as product of sum since every sum term are multiplied in this form.

For eg. $f(x,y,z)=x.(y+z).(x'+y)$ is a CNF.

Eg. 6 **Express** $(x+y+z)(xy+x'z)'$ **in CNF.**

Sol. $(x+y+z)(xy+x'z)'$

$= (x+y+z)((xy)'.(x'z)')$ (Using Demorgan's Law)

$= (x+y+z)(x'+y')(x+z')$ (Using Demorgan's Law)

Principle Conjunctive Normal Form (PCNF)

If in a CNF every term contain every variable (either in complemented or uncomplemented form). This is known as Principle Conjunctive Normal Form or standard product of sum (SPOS). Every term in SPOS is known as maxterm.

Conversion of CNF into PCNF:

To convert CNF into PCNF first we find the term with missing variable and add the product of missing variable and simply it and finally drop the common term.

Eg. **Convert** $(x+y+z)(x'+y')(x+z')$ **into PCNF.**

Sol: $f(x,y,z)=(x+y+z)(x'+y')(x+z')$

$= (x+y+z)(x'+y'+zz')(x+z'+yy')$

$= (x+y+z)(x'+y'+z)(x'+y'+z')(x+y+z')(x+y'+z')$

Application of Boolean Algebra

Boolean Algebra is used in various fields. One important application is switching system i.e. on-off electric switch which may be in two states on or off. (closed or open).

In first state current does not flow but in second state current flow in circuit. So a switch is said to be open if the current does not flow and it is said to be closed if the current flows.

The switches are connected in two ways:

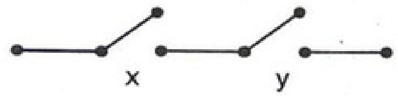
(i) In series

(ii) In parallel

(i) In series : Two switches are said to be connected in series if current flows when if and only if both switches are closed (on) otherwise the current does not flow.

This is equivalent to 'logical AND' operation of Boolean Algebra.

Let's consider two switches x and y are connected in series as shown in figure.



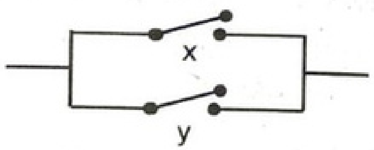
Now 0 denotes open and 1 denotes closed then result can be represented as :

x	y	Output
open (0)	open(0)	off(0)
open (0)	closed (1)	off(0)
closed (1)	open(0)	off(0)
closed (1)	closed (1)	on (1)

(ii) In Parallel : Two switches are said to be in parallel, if the current does not flow if both switches are open (off).

This is equivalent to logical OR operation in Boolean Algebra.

Let's consider two switches x and y are connected in parallel as shown in figure



Now 0 denotes open and 1 denotes closed, the result can be represented as :

x	y	Output
open (0)	open(0)	off(0)
open (0)	closed (1)	on (1)
closed (1)	open(0)	on (1)
closed (1)	closed (1)	on (1)

Questions

Very Short Type Questions

1. Define Boolean Algebra.
2. Write the Demorgan's law.
3. Write the absorption law.
4. a + ab + c = ?
5. Define minterm and maxterm.
6. Write the example of DNF and CNF.
7. Define PCNF and PDNF.
8. What is switching system.

Short Type Questions

1. Write the absorption law and prove it.
2. What is Boolean function?
3. Define switching system and its series and parallel interconnection.
4. Define the following logic gates

   (i) OR (ii) AND (iii) NOR
5. Prove that $(a'+c)(a+b)=a'b+ac$
6. Prove that $ab+\left[c(a'+b')\right]=ab+c$

Long Type Questions

1. Prove that the following boolean identites–

   (i) $(a \cdot b \cdot c) + a \cdot b = a \cdot b$

   (ii) $(a \cdot b' + b \cdot c) \cdot (a \cdot c + b \cdot c') = a \cdot c$

   (iii) $(a + b') \cdot (b + c') \cdot (c + a') = (a' + b) \cdot (b' + c) \cdot (c' + a)$

   (iv) $a \cdot b + a' \cdot b' = (a' + b) \cdot (a + b')$

   (v) $[\{(a' \cdot b')' + c\} \cdot (a + c)]' = a' \cdot c'$

2. Simplity the following Boolean expressions–

   (i) $a \cdot b + (a \cdot b' \cdot c) + b \cdot c$

   (ii) $a \cdot b + (a' \cdot b \cdot c) + b \cdot c$

   (iii) $[(a' \cdot b')' + a] \cdot (a + b')'$

3. Factorise the following Boolean expressions in the Boolean algebra B–

   (i) $a + a' \cdot b \cdot c$

   (ii) $a \cdot b' + a' \cdot b$

   (iii) $a \cdot b + b \cdot c + c \cdot a$

1. Express the following Boolean Function in D.N.F.

   (i) $[x_1 + x_2' + (x_2 + x_3')]' + x_2 \cdot x_3$

   (ii) $(x_1 + x_2) \cdot (x_1' + x_3')$

   (iii) $(x_1 + x_2) \cdot (x_1 + x_2') \cdot (x_1' + x_3)$

   (iv) $(x \cdot y + y' \cdot z)' + z'$

2. Express the following Boolean function in C.N.F.

   (i) $x \cdot y + x \cdot y' + x \cdot z$

   (ii) $x \cdot (y + z) + x \cdot (y + z')$

   (iii) $(x + y) \cdot (x' + y')$

   (iv) $(x_1 + x_2) \cdot (x_1' + x_2') \cdot x_1$

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