UNIT-I

Number Systems: Number systems- natural numbers, integers, rational numbers, real numbers, complex numbers, arithmetic modulo a positive integer. Radix r representation (decimal and binary), Change of radix(decimal to binary and vice versa). Binomial Theorem and Mathematical Induction : Binomial theorem for positive integral indices, general and middle term in binomial expansion with simple applications. Some simple problems of Principle of Mathematical induction.

Recurrence Relations and Generating Functions : Recurrence relation, linear recurrence relation with constant coefficients, solution of linear recurrence relation with constant coefficients. Generating functions, Solution of recurrence relations using generating functions.

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NUMBER SYSTEM

Introduction:-

Number system deals with number and their representation in different system. We can distinguish these numbers using Base of the system.

In day to day life we use decimal number system but our computer deal various number system like binary number system, octal system & hexadecimal number system. These number system are widely used in digital system like logic circuits, microprocessor etc.

Natural Number: The number system which can be used to count no. of objects is known as natural number. It is denoted by N.

N= {1, 2, 3......}

Whole Number: If we include 0 in natural number system. It is known as whole numbers denoted by W

W= {0, 1, 2, 3...}

Integers: The number system includes natural number, their negative & zero is known as integers denoted by Z or I.

Z or I = {.... -2, -1, 0, 1, 2....}

Rational numbers: The numbers which can be denoted by $\frac{p}{q}$ form, where $q \neq 0$ is known as Rational number. This system can be denoted by Q

$$Q = \left\{\frac{p}{q}; q \neq 0\right\}$$

Eg. $-\frac{2}{3}, \frac{5}{7}$

Irrational Numbers: The numbers which cannot be denoted by form are known as irrational numbers.

Eg., $\sqrt{2}$, $\sqrt{3}$

Real Numbers: Real numbers include rational & irrational number. It is denoted by R.

So R = (Rational number) U (Irrational number)

Complex Numbers: A number is known as complex number if it contains imaginary part as well as real part. So we can say that the complex number is combination of real & imaginary part. It is denoted by

$z = x + \imath y$

Where x is real part & y is imaginary part.

Arithmetic Modulo a Positive Integer

Let's consider a, b, & n are integer then the statement like $a \cong b$ (mod n) is arithmetic module which can be used as a is congruent to b module n $a \cong b$ (mod n) is equivalent to statement.

"a – b is divisible by n".

Eg. $28 \cong 5$(Modulo 5)

= 28-3 = 25

& 25 is divisible by 5.

Types of the Number System:

The positional numbers are of four types. Positional number are the number whose value depends on position of the digit i.e. if we change the position of the digit the value of the number will be changed. These types are:

- Decimal system
- Binary system
- Octal system
- Hexadecimal system

Here we study only about decimal & binary numbers system & their conversion.

Decimal System:

It is a positional number system with base (or radix) 10, so this system uses ten symbols or digits like 0, 1, 2, 3 ……9.

The successive position from left to right of the decimal number are units, tens, hundreds, thousands digit.

For e.g. In 552, 2 is in units position 5 is in tens position & 5 is in hundreds position. i.e.

In 552, the digit 2 means $2 \times 10^0 = 2$

The digit 5 means $5 \times 10^1 = 50$

The digit 5 means $5 \times 10^2 = 500$

Binary Number system:

It is a positional number with base 2 so this system uses two symbols or digits like 0 & 1.

Each Position in a binary number represents 0 powers of the base 2.

For Eg. If we represent a number $(1001)_2$ this will be equivalent to $1\times 2^3 + 0\times 2^2 + 0\times 2^1 + 1\times 2^0 = 9$

Conversion of decimal to binary form:

A decimal number may have two parts integers or fractional or both. We use different mechanism for integer part & fractional part.

Integer Part: For integer part we use following steps

Steps 1: Divide the decimal integer by 2.

Step 2: Write the remainder obtained by step 1.

Step 3: Now again the quotient by 2.

Step 4: Now again write the remainder obtain in step 3.

Step 5: Repeat step 3 & 4 until the quotient became zero.

Eg. $(236)_{10} = (?)_2$

Eg(1)

		Reminder
2	236	
2	118	0
2	59	0
2	29	1
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

So $(236)_{10} = (11101100)_2$

Eg. 2 Convert 201 into binary form

2	201	Reminder
2	100	1
2	50	0
2	25	0
2	12	1
2	6	0
2	3	0
2	1	1
	0	1

So $(201)_{10} = (11001001)_2$

Eg. 3 Show that $(38)_{10} + (69)_{10} = (1101011)_2$ **[RU BCA 2005]**

Solution: $(38)_{10} + (69)_{10} = (107)_{10}$

2	107	Reminder
2	53	1
2	26	1
2	13	0
2	6	1
2	3	0
2	1	1
	0	1

So $(107)_{10} = (1101011)_2$

Eg. 4: Find the value of $(195)_{10} + (105)_{10}$ in binary number. [R.U. B.CA 2006]

Solution: 195+105 =300

So,

2	300	Reminder
2	150	0
2	75	0
2	37	1
2	18	1
2	9	0
2	4	1
2	2	0
2	1	0
	0	1

So $(195)_{10} + (105)_{10} = (300)_{10} = (100101100)_2$

Fractional Part: For fractional part we use following steps: -

Step 1: We first multiply the fractional part by 2

Step 2: The integer part of the result will be either 1 or 0. We write the integer part aside & multiply the fractional part by 2.

Step 3: Again we write the integer part aside & contain with step 2, untill we obtain zero in fractional part.

E.g. 5 Convert $(0.625)_{10}$ into binary form.

0.625 ×2	
1.250 ×2	1
0.500 ×2	0
1.000	1

So $(0.625)_{10} = (0.101)_2$

Eg. 6. $(0.875)_{10} = (?)_2$

0.875 × 2	
1.750 ×2	1
1.500 ×2	1
1.000	1

So $(0.875)_{10} = (0.111)_2$

Mixed Problems:

Eg. 7 $(78.65625)_{10}$ = $(?)_2$

Solution: Integer Part

2	78	Reminder
2	39	0
2	19	1
2	9	1
2	4	1
2	2	0
2	1	0
	0	1

So $(78)_{10} = (1001110)_2$

Fractional part:

0.65625 ×2	
1.31250 ×2	1
0.62500 ×2	0
1.25000 ×2	1
0.50000 ×2	0
1.00000	1

$(0.65625)_{10} = (0.10101)_2$

Hence $(78.65625)_{10} = (1001110.10101)_2$

Eg. 8. $(201.2)_{10}$ = $(?)_2$

Solution: **Integer part:**

2	201	Reminder
2	100	1
2	50	0
2	25	0
2	12	1
2	6	0
2	3	0
2	1	1
	0	1

$(201)_{10} = (11001001)_2$

Fraction part:

.2 × 2	
0.4 ×2	0
0.8 ×2	0
1.6 ×2	1
1.2 ×2	1

So $(.2)_{10} = (0.00110011\ldots)_2$

So $(201.2)_{10} = (11001001.00110011\ldots)_2$

Conversion of Binary into decimal form

Integer form: To Convert binary to decimal we follow the following steps.

Step 1: Multiply each digit from right to left by 2's power i.e. $2^0, 2^1, 2^3 \ldots$

Step 2: Solve each term & add them

Eg. 9. $(110101)_2 = (?)_{10}$

Solution: $110101 = (1 \times 2^5) + (1 \times 2^4) + (0 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$

$= 32+16+0+4+0+1$

$= 53$

Eg. 10. Convert $(100110011)_2$ into decimal form. **[RU BCA 2006]**

$(100110011)_2 = (1\times 2^8) + (0\times 2^7) + (0\times 2^6) + (1\times 2^5) + (1\times 2^4) + (0\times 2^3) + (0\times 2^2) + (1\times 2^1) + (1\times 2^0)$

$= 256+0+0+32+016+0+0+2+1$

$= 307$

Fractional part: For fractional part use the following steps.

Step 1: Multiply each digit form left to right by 2's negative power i.e. $2^{-1}, 2^{-2}, 2^{-3}\ldots$

Step 2: Simply each term & find the result by adding them.

Eg. 11. $(0.100001)_2 = (?)_{10}$

Solution: $(0.100001)_2 = (1\times 2^{-1}) + (0\times 2^{-2}) + (0\times 2^{-3}) + (0\times 2^{-4}) + (0\times 2^{-5}) + (1\times 2^{-6})$

$= 1\times 0.5+0+0+0+0+1\times 0.15625$

$= (.515625)_{10}$

Eg. 12. Convert $(0.101010)_2$ into decimal form.

Solution: $(0.101010)_2 = (1\times 2^{-1}) + (0\times 2^{-2}) + (1\times 2^{-3}) + (0\times 2^{-4}) + (1\times 2^{-5}) + (0\times 2^{-6})$

$= 1\times 0.5+0+1\times 0.125+0+1\times 0.3125+0$

$= (0.65625)_{10}$

Hence $(0.101010)_2 = (0.65625)_{10}$

Mixed problem:

Eg.13. $(101.110)_2 = (?)_{10}$

Solution: Integer part-

$101 = (1\times 2^2) + (0\times 2^1) + (1\times 2^0)$

$= 4+0+1 = 5$

Fractional part:

$(.110)_2 = (1\times 2^{-1}) + (1\times 2^{-2}) + (0\times 2^{-3})$

$= 0.5+0.25+0$

$= 0.75$

Hence $(101.110)_2 = (5.75)_{10}$.

Eg. 14. Convert $(111001.1101)_2$ into decimal form.

Solution: $(111001.1101)_2$

Integer part:

$(111001)_2 = (1\times 2^5) + (1\times 2^4) + (1\times 2^3) + (0\times 2^2) + (0\times 2^1) + (1\times 2^0)$

$= 32+16+8+0+0+1$

$= 57$

Fractional part:

$(0.1101)_2 = (1\times 2^{-1}) + (1\times 2^{-2}) + (0\times 2^{-3}) + (1\times 2^{-4})$

$= .5+.25+0+0.0625$

$= 0.8125$

Hence $(111001.1101)_2 = (57.8125)_{10}$

Questions

Very Short Type:

1. Define Integers.
2. Define Real Numbers.
3. What is Complex Number?
4. What is natural numbers?

Short Type:

1. Convert the number $(111001.1101)_2$ into decimal form. **[Raj. Univ. BCA 2003]**
2. Convert the decimal 21.6875 into a binary number. **[Raj Univ. BCA 2004]**
3. Convert $(111001100)_2$ into decimal form. **[Raj Univ. BCA 2006]**
4. Convert the decimal number 21.6875 into a binary number. **[Raj Univ. BCA 2004]**
5. Convert the decimal 43.375 into a binary equivalent. **[Raj Univ. BCA 2007]**
6. Convert $(0.65625)_{10}$ into binary system. **[Raj Univ. BCA 2009]**

Long Type:

1. Convert the following decimal into binary.

   (a) 278 (b) 79

   (c) 86.56 (d) 92.625

   (e) 0.205

2. Convert the following binary into decimal.

   (a) 101.1011

   (b) 0.10101011

   (c) 110110101

   (d) 101.1011

   (e) 0.1010

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MATHEMATICAL INDUCTION

Introduction

The mathematical induction is a technique to prove the mathematical statement (Theorems or identities) for natural number.

FIRST PRINCIPLE OF MATHMETICAL INDUCTION

We follow the following algorithm to prove that a statement is true for all natural numbers using first principle of mathematical induction.

- Obtain *p (n)* by equating the given statement to prove.
- Prove that p (n) is true for $n=1$.
- Assume that *p (m)* is true.
- Using assumption in previous step, prove that $p(m+1)$ is true.
- Now conclude by the first principle of mathematical induction that *p (n)* is *true for all* nN.

Eg.1 Prove that $1+2+3+.....+n = \frac{n(n+1)}{2}$

Sol: Let $p(n) = 1+2+3+......+n = \frac{n(n+1)}{2}$ (i)

For n = 1

We get 1 from LHS

& by RHS, We obtain

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

$\Rightarrow$ p (*1*) id true.

Let the theorem is true for *p (m)*

So $1+2+3+\ldots\ldots m = \dfrac{m(m+1)}{2}$ (ii)

Now we have to prove *p (m+1)* is true.

So $1+2+3\ldots\ldots m + (m+1)$

$= \dfrac{m(m+1)}{2} + (m+1)$ [using (ii)]

$= \dfrac{m(m+1)+(2m+2)}{2}$

$= \dfrac{m(m+1)+2(m+1)}{2}$

$= \dfrac{(m+1)(m+2)}{2}$

$\Rightarrow$ *P (m+1)* is true.

$\Rightarrow$ Theorem is true $\forall$N.

Eg.2 Prove by the principle of mathematical induction

$P(n): 1+2+2^2+2^3+\ldots+2^n = 2^{n+1}-1$ (*for* $n \geq 0$).

Sol: Let $P(n): 1+2+2^2+2^3+\ldots\ldots +2^n = 2^{n+1}-1$ (i)

For *n = 0*

We get 1 form LHS

& by RHS, We obtain

$2^{0+1}-1 = 2^{0+1}-1 = 2^1-1 = 1$

$\Rightarrow$ *P (1)* is true.

Let the theorem is true for P (m) i.e.

$1+2+2^1+2^3+\ldots\ldots + 2^m = 2^{m+1}-1$ (ii)

Now we have to prove P (m+1) is true.

$1+2+2^1+2^3+\ldots\ldots + 2^m \times 2^{m+1}$

$= 2^{m+1}-1+2^{m+1}$ [using (ii)]

$= 2.2^{m+1}-1$

$= 2^{m+2}-1$

$2^{(m+1)+1}-1$

$\Rightarrow$ P (m+1) is true.

$\Rightarrow$ Theorem is true $\forall$N.

Eg.3 Prove by the principle of mathematical induction that: ***n (n+1)(2n+1)*** **is divisible by 6 for all n$\in$N.**

Sol: Let *P (n)* be the statement "*n (n+1) (2n+1)* is divisible by 6".

For n = 1

We have *P (1): 1(1+1) (2+1)*

= 6, which is divisible by 6

So, P (1) is true.

Let *P (m)* is true. Then *m (m+1) (2m+1)* = 6λ, for some $\lambda \in$ N (i)

Now, we shall show that *P (m+1)* is true. Now,

$(m+1)\,[(m+1)+1)]\,[2(m+1)+1]$

$= (m+1)(m+2)\,[(2m+1)+2]$

$= (m+1)(m+2)(2m+1)+2(m+1)(m+2)$

$= m(m+1)(2m+1)+2(m+1)(2m+1)+2(m+1)(m+2)$

$= m(m+1)(2m+1)+2(m+1)(2m+1+m+2)$

$= m(m+1)(2m+1)+2(m+1)(3m+3)$

$= m(m+1)(2m+1)+6(m+1)^2 = 6\lambda + 6(m+1)^2$

$= 6\{\lambda + (m+1)^2\}$, which is divisible by 6

$\Rightarrow$ *P (m+1)* is true.

Hence, by the principle of mathematical induction, the given statement is true for all n $\in$ N.

Eg.4 Prove by the principle of mathematical induction:

$$P(n): \frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\frac{1}{2^n}=1-\frac{1}{2^n}$$

Sol: Let $P(n)=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\frac{1}{2^n}=1-\frac{1}{2^n}$

For n = 1

We get $\frac{1}{2}$ From LHS

& by RHS, We obtain

$$1-\frac{1}{2^n}=1-\frac{1}{2^1}=1-\frac{1}{2}=\frac{1}{2}$$

$\Rightarrow$ *P (1)* is true.

Let the theorem is true for P (m), so

$$P(n)=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\frac{1}{2^m}=1-\frac{1}{2^m}$$

Now we have to prove *p (m+1)* is true

So $P(m+1)=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^m}+\frac{1}{2^{m+1}}$

$$=1-\frac{1}{2^m}+\frac{1}{2^{m+1}} \quad \text{[using (i)]}$$

$$=\frac{2^{m+1}-2+1}{2^{m+1}}$$

$$=\frac{2^{m+1}-1}{2^{m+1}}$$

$$=\frac{2^{m+1}}{2^{m+1}}-\frac{1}{2^{m+1}}$$

$$=1-\frac{1}{2^{m+1}}$$

$\Rightarrow$ *P (m+1)* is true

$\Rightarrow$ Theorem is true for all N.

Eg. Prove by the principle of mathematical induction. The given statement is true for all n$\in$N.

$$\mathbf{1+4+7+\ldots+(3n-2)=\frac{1}{2}n\,(3n-1)}$$

Sol: Let *P (n)* be the statement given by

$$P(n): 1+4+7+\ldots+(3n-2)=n\,(3n-1)$$

For n = 1

We have $P(1): \frac{1}{2}\times(1)\times(3\times1-1)$

$$\because 1=\frac{1}{2}\times(1)\times(3\times1-1)$$

So, *P (1)* is true

Let P (m) be true. Then,

$$1+4+7+\ldots+(3m-2)=\frac{1}{2}m(3m-1) \quad \ldots\text{(i)}$$

Now we have to prove *p (m+1)* is true

$$1+4+7+\ldots+(3m-2)+[3(m+1)-2]=\frac{1}{2}(m+1)(3(m+1)-1)$$

Now

$$1+4+7+\ldots+(3m-2)+[3(m+1)-2] = \frac{1}{2}m(3m-1)+[3(m+1)-2]$$

$$= \frac{1}{2}m(3m-1)+(3m+1) = \frac{1}{2}[3m^2-m+6m+2]$$

$$= \frac{1}{2}[3m^2+5m+2] = \frac{1}{2}(m+1)(3m+2) = \frac{1}{2}(m+1)[3(m+1)-1]$$

So, *P (m+1)* is true.

So, by the principle of mathematical induction, the given theorem is true for all nN.

Eg. Prove by induction that the sum of the cubes of three consecutive integers is divisible by 9.

Sol: Let the three consecutive numbers are n, $n+1$ and $n+2$.

$P(n) = n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9

For n = 1

$$P(1) = 1+8+27 = 36 \qquad \text{(i)}$$

Which is divisible by 9?

So, *P (1)* is true.

Let P (m) be true. Then,

$$P(m) = m^3 + (m+1)^3 + (m+2)^3 = 9\lambda \qquad \text{(ii)}$$

Now we have to prove p (m+1) is true

$$P(m+1) = (m+1)^3 + (m+2)^3 + (m+3)^3$$

$$= m^3 + (m+1)^3 + (m+2)^3 + (9m^2+27m+27)$$

$$= 9\lambda + 9(m^2+3m+3) = 9(\lambda + m^2+3m+3) \qquad \text{(using (ii))}$$

$$= 9\beta, \text{ where } \beta = (\lambda + m^2+3m+3)$$

Which is divisible by 9

So, *P (m+1)* is true

So, by the principle of mathematical induction, the given statement is true for all $n \in N$.

Eg. Show that n^3+2n is divisible by 3.

For $n = 1$

$$P(1) = 1^3+2 \times 1 = 3 \qquad \text{....(i)}$$

Which is divisible by 3?

So, *P (1)* is true

Let *P (m)* be true. Then,

$P(m) = m^3 + 2m$ is divisible by 3.

i.e. $m^3+2m = 3\lambda$ ....(ii)

Now we have to prove *p (m+1)* is true

$$P(m+1) = (m+1)^3+2(m+1) = m^3+1+3m(m+1)+2m+2$$

$$= m^3+1+3m^2+3m+2m+2$$

$$= m^3+3+5m+3m^2 = m^3+2m+3+3m+3m^2$$

$$= 3\lambda+3+3m+3m^2 = 3\lambda+3m^2+3m+3 \qquad \text{(using (ii))}$$

$= 3(\lambda+m^2+m+1)$, which is divisible by 3.

So, P (m+1) is true

So, by the principle of mathematical induction, the given statement is true for all nN.

Questions

Very Short Type Questions:

1. What is mathematical induction?
2. Prove the identity for first natural number

$$1^3+2^3+3^3+\ldots+n^3 = \left(\frac{n(n+)}{2}\right)^2$$

Short Type Questions:

1. Use mathematical induction to show that

$$1^2+2^2+ \dots\dots\dots n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Use mathematical induction to show that

$$1^3+2^3+3^3+ \dots + n^3 = \left(\frac{n(n+)}{2}\right)^2$$

3. Let p the proposition that the sum of the first n odd numbers is n^2. Prove it by using the mathematical induction principle.

4. Prove by mathematical induction to show that $n^2 + n$ is an even natural number.

Long Type Questions:

1. Use mathematical induction to show that

$2+2^2+2^3+ \dots\dots + 2^n = 2^{n+1} -2 \quad \forall n \in N$ **(Raj. Univ. BCA 2007)**

2. Using first principle of mathematical induction, prove that $3^{2n} + 7$ is divisible by 8 for all $n \in N$.

3. Prove by the principle of mathematical induction that 9^n-8n-1 is divisible by 64, for all integers $n \geq 2$.

4. Prove by mathematical induction that:

$P(n) = 1.2+2.2^2+ \dots\dots n.2^n = (n-1)\, 2^{n+1}+2$; for $n \in N$.

5. Prove that n (n+1) (n+2) is a multiple of 6 using first principle of mathematical induction.

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BINOMIAL THEOREM

Introduction

We know that $(x+y)^2 = x^2+2xy+y^2$. To find $(x+y)^3$ we again multiply $(x+y)^2$ by $(x+y)$.

i.e. $(x+y)^3 = (x+y)^2(x+y)$

$$= x^3+3x^2y+3xy^2+y^3$$

The sum $x^3+3x^2y+3xy^2+y^3$ is called binomial expansion of $(x+y)^3$. If we again multiply $(x+y)^3$ by $(x+y)$, we get the binomial expansion of $(x+y)^4$.

Consider the following binomial expansions :

$$(x+y)^0 = 1$$

$$(x+y) = x+y$$

$$(x+y)^2 = x^2+2xy+y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$$

$$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$$

$$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$$

The exponent on the variable x are decreasing whereas the exponents on the variable y are increasing as we read from left to right. The sum of the exponents in each term is the same for that entire line.

Obtaining the Coefficients

If we write out only the coeffecients of the expansions, we easily see a pattern. This triangular array of coeffecients for the binomial expansion is called Pascal's triangle.

1	$(x+y)^0 = 1$
1 1	$(x+y) = 1x+1y$
1 2 1	$(x+y)^2 = 1x^2+2xy+1y^2$
1 3 3 1	$(x+y)^3 = 1x^3+3x^2y+3xy^2+1y^3$
1 4 6 4 1	$(x+y)^4 = 1x^4+4x^3y+6x^2y^2+4xy^3+1y^4$

The Binomial Theroem

If we wanted to expand a binomial expression with a large power eg. $(1+x)^{30}$, use of Pascal's triangle would not be recommended because of the need to generate a large number of rows of the triangle. So we use Binomial Theorem. It can be defined as :

$$(x+a)^n = {}^nC_0x^na^0 + {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 + ... + {}^nC_rx^{n-r}a^r + ... + {}^nC_nx^0a^n$$

Some more form of Binomial Theorem :

$$(x-a)^n = {}^nC_0x^na^0 - {}^nC_1x^{n-1}a^1 + {}^nC_2x^{n-2}a^2 - {}^nC_3x^{n-3}a^3 + ... + (-1)^n\,{}^nC_nx^0a^n$$

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$$

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + ... + (-1)^n\,{}^nC_nx^n$$

$$(x+a)^n + (x-a)^n = 2\left[{}^nC_0x^na^0 + {}^nC_2x^{n-2}a^2 + ...\right]$$

$$(x+a)^n - (x-a)^n = 2\left[{}^nC_1x^{n-1}a^1 + {}^nC_3x^{n-3}a^3 + ...\right]$$

The points to Remember :

- The coefficient of $(r+1)^{th}$ term is ${}^nC_r$ in expansion of $(1+x)^n$.

- The coefficient of x^r is ${}^nC_r$ in expansion of $(1+x)^n$.
- If n is odd then $\{(x+a)^n + (x-a)^n\}$ and $\{(x+a)^n - (x-a)^n\}$ both have $\left(\frac{n+1}{2}\right)$ terms.
- If n is even then $\{(x+a)^n + (x-a)^n\}$ has $\left(\frac{n}{2}+1\right)$ terms.
- If n is even then $\{(x+a)^n - (x-a)^n\}$ has $\left(\frac{n}{2}\right)$ terms.

Ex. 1. Expand $(x+2a)^5$ by binomial theorem.

Sol. $(x+2a)^5 = {}^5C_0(x)^5(2a)^0 + {}^5C_1x^4(2a)^1$

$+{}^5C_2x^3(2a)^2 + {}^5C_3x^2(2a)^3 + {}^5C_4x\,(2a)^4 + {}^5C_5(x)^0(2a)^5$

$= x^5 + 5x^4(2a) + 10x^3(4a^2) + 10x^2(8a^3) + 5x\,(16a^4) + 32a^5$

$= x^5 + 10x^4 + 40x^3a^2 + 80x^2a^3 + 80xa^4 + 32a^5$

Ex 2. Write the binomial expansion of $(x^2-2a)^5$

Sol. $(x^2-2a)^5 = {}^5C_0(x^2)^5(2a)^0 - {}^5C_1(x^2)^4(2a)^1 + {}^5C_2(x^2)^3(2a)^2$

$-{}^5C_3(x^2)^2(2a)^3 + {}^5C_4(x^2)\,(2a)^4 - {}^5C_5(x^2)^0(2a)^5$

$= x^{10} - 10x^8a + 40x^6a^2 - 80x^4a^3 + 80x^2a^2 - 32a^5$

Ex 3. Find $(a+b+c)^4$

Sol. Let us consider $b+c=k$

Then, $(a+b+c)^4=(a+k)^4$

$(a+k)^4={}^4C_0a^4k^0+{}^4C_1a^3k^1+{}^4C_2a^2k^2+{}^4C_3a^1k^3+{}^4C_4a^0k^4$

$=a^4+4a^3k+6a^2k^2+4ak^3+k^4$

Now using $k=b+c$

$=a^4+4a^3(b+c)+6a^2(b+c)^2+4a(b+c)^3+(b+c)^4$

$=a^4+4a^3b+4a^3c+6a^2b^2+6a^2c^2+12a^2bc$

$+4a\left(b^3+3bc^2+3b^2c+c^3\right)+\left(b^4+4b^3c+6b^2c^2+4bc^3+c^4\right)$

$=a^4+b^4+c^4+4a^3b+4a^3c+6a^2b^2$

$+6a^2c^2+12a^2bc+4ab^3+12abc^2+12ab^2c+4ac^3+4b^3c+6b^2c^2+4bc^3$

Ex 5. Expand $\left(x+\frac{1}{x}\right)^4$

Sol. Expanding by Binomial Theorem

$$\left(x+\frac{1}{x}\right)^4={}^4C_0(x)^4\left(\frac{1}{x}\right)^0+{}^4C_1(x)^3\left(\frac{1}{x}\right)^1+{}^4C_2(x)^2\left(\frac{1}{x}\right)^2$$

$$+{}^4C_3(x)\left(\frac{1}{x}\right)^3+{}^4C_4(x)^0\left(\frac{1}{x}\right)^4$$

$=x^4+4x^2+6+\frac{4}{x^2}+\frac{1}{x^4}$

Ex 6. Find the value of (a) $(x+5)^5+(x-5)^5$

(b) $(x+5)^5-(x-5)^5$

Sol. $(x+5)^5+(x-5)^5$

$\because (x+a)^n+(x-a)^n=2\left[{}^nC_0x^na^0+{}^nC_2x^{n-2}a^2+....\right]$

$(x+5)^5+(x-5)^5=2\left[{}^5C_0x^5+{}^5C_2x^3\times5^2+{}^5C_4x\times5^4\right]$

$=2\left[x^5+250x^3+3125x\right]$

Now

$(x+5)^5-(x-5)^5$

$\because (x+a)^n-(x-a)^n=2\left[{}^nC_1x^{n-1}a^1+{}^nC_3x^{n-3}a^3+....\right]$

So, $(x+5)^5-(x-5)^5=2\left[{}^5C_1x^4\times5+{}^5C_3x^2\times5^3+{}^5C_5x^0\times5^5\right]$

$=2\left[25x^4+1250x^2+3125\right]$

General Term and Middle Term in a Binomial Expansion:

Let us consider the $(r+1)^{th}$ term denoted by T_{r+1} then

- In binomial expansion of $(x+a)^n$, the $(r+1)^{th}$ term is given by

$$T_{r+1}={}^nC_rx^{n-r}a^r$$

- In binomial expansion of $(x-a)^n$, the $(r+1)^{th}$ term is given by

$$T_{r+1}=(-1)^r\,{}^nC_rx^{n-r}a^r$$

- In binomial expansion of $(1+x)^n$, the $(r+1)^{th}$ term is given by

$$T_{r+1} = {}^nC_r x^r$$

- In binomial expansion of $(1-x)^n$, the $(r+1)^{th}$ term is given by

$$T_{r+1} = (-1)^r\, {}^nC_r x^r$$

Middle Term

- In case of n is even (i.e. number of terms are odd) then the middle term will be $\left(\frac{n}{2}+1\right)^{th}$ term.
- In case n is odd, then $\left(\frac{n+1}{2}\right)^{th}$ and $\left(\frac{n+3}{2}\right)^{th}$ terms are the two middle terms.

Ex 7. Find the general term and middle term in the expression $(3x-4)^6$

Sol. General Term T_{r+1} of $(x+a)^n = {}^nC_r x^{n-r} a^r$

So general term of $(3x-4)^6$

$$T_{r+1} = {}^6C_r (3x)^{6-r} (-4)^r \quad \text{...(1)}$$

Middle term of $(3x-4)^6 = \left(\frac{n}{2}+1\right)^{th}$

$$= \left(\frac{6}{2}+1\right)^{th}$$

$$= 4^{th}$$

In equation (1), $r = 3$

$$T_4 = T_{3+1} = {}^6C_3 (3x)^3 (-4)^3$$

$$= -34560x^3$$

Ex 8. Find the fourth term of $(x+y)^{12}$.

Sol. Since general term of $(x+a)^n$

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

$$T_4 = T_{3+1} = {}^{12}C_3 x^9 y^3$$

$$= 220x^9y^3$$

Ex 9. Find the sixth term of $(x^2-2y)^7$.

Sol. Since general term of $(x-a)^n$

$$T_{r+1} = (-1)^r\, {}^nC_r x^{n-r} a^r$$

So sixth term of $(x^2-2y)^7$

$$T_6 = T_{5+1} = (-1)^5\, {}^7C_5 (x^2)^{7-5} (2y)^5$$

$$= -672x^4y^5$$

Questions

Very Short Type Questions

Q1. How many terms are there in expansion of $(a+b)^4$.

Q2. Write the fourth term of $(x+3)^5$.

Q3. Write the expansion of $(x+a)^n$.

Q4. Write the expansion of $(x-a)^n$.

Q5. Write the middle term of $(x+3)^4$.

Q6. Write the middle term of $(x+3)^5$.

Short Type Questions

Q1. Write the binomial expansion of $\left(x^2-2a\right)^4$.

Q2. Find the binomial expansion of $\left(2x-\frac{3}{x}\right)^7$.

Q3. Find the term not containing x in the expresssion of $\left(2x+\frac{1}{x}\right)^{12}$.

Q4. Expand $(3x-2y)^4$.

Q5. Expand $(x+2a)^5$.

Long Type Questions

Q1. Expand

(i) $\left(1+x+x^2\right)^3$ (ii) $\left(1-x-x^2\right)^3$

Q2. Expand

(i) $(x-3)^6+(x+3)^6$ (ii) $(x-3)^6-(x+3)^6$

Q3. Find the 9th term of $\left(\frac{x}{y}-\frac{3y}{x}\right)^{10}$.

Q4. Find the general term and middle term of $\left(x+\frac{y}{a}\right)^9$.

4

RECURRENCE RELATION AND GENERATING FUNCTIONS

Introduction

A recurrence relation define a sequence by r^{th} value in term of $(r-1)^{th}$ value with given initial values about the beginning of the sequence.

In this chapter we will study about the recurrence relation, linear recurrence relation with constant coeffecients, solution of recurrence relations using generating functions etc.

Recurrence Relation :

An equation (or formula) for a numeric function which connect or established the relationship between a_r and its predecessor $(a_0, a_1, ... a_r)$ for a given r. The initial conditions for the sequence $(a_0, a_1, ... a_r)$ are given values for a finite number of terms of the sequence.

For eg. $a_r = 4a_{r-1}, a_0 = 1$ is the recurrence relation.

The recurrence relation is also known as difference equations.

Linear Recurrence Relation with constant coeffecients :

A recurrence relation in the form

$$C_0 a_r + C_1 a_{r-1} + ... + C_k a_{r-k} = f(r)$$

is known as linear recurrence relation with constant coeffecients where C_i is constant. If C_0 and C_k both are non zero then recurrence relation is known as recurrence relation of order K.

For eg. $3a_r + 5a_{r-1} = 2$

is a linear recurrence relation of first order.

Homogeneous Solution :

Let us consider a is a numeric function then recurrence relation

$$C_0 a_r + C_1 a_{r-1} + ... + C_k a_{r-k} = 0$$

is known as homogeneous recurrence relation of order K.

The solution of homogeneous recurrence relation of constant coeffecient is in form of $a_r = A\alpha_i^r$, where α_i is known as characteristic root and A is a constant.

So, we get

$$C_0\alpha^k + C_1\alpha^{k-1} + ... + C_k = 0$$

is known as characteristic equation. These are following cases:

Case 1 : When the roots of characteristic equation is real but different, then complementary function (C.F.)

$$A_1\alpha_1^r + A_2\alpha_2^r + ... + A_k\alpha_k^n$$

is the solution of homogeneous equation.

Case 2 : When some of the roots of characteristic equation are real and identical:

In this condition, the complementary function

$$(A_1 + A_2)\alpha_1^r + \left(A_3 + A_4A_5r^2\right)\alpha_2^r$$

where $A_1 = A_2$ and $A_3 = A_4 = A_5$

Case 3 : When two roots of the characteristic equation are complex and distinct.

Let's consider complex roots are $\alpha + i\beta$ and $\alpha - i\beta$, then solution is

$$A(\alpha + i\beta)^r + B(\alpha - i\beta)^r$$

Case 4: When roots of the characteristic equation are complex and identical.

If $\alpha + i\beta$ is identical.

$$\rho^r (A_1 + A_2r)\cos r\theta + \rho^r (A_3 + A_4r)\sin r\theta$$

$$\rho = \sqrt{\alpha^2 + \beta^2} \text{ and } \tan\theta = \frac{\beta}{\alpha}$$

Eg 1. If $a_r = a_{r-1} + a_{r-2}$, $r \geq 2$ where $a_0 = 0$ and $a_1 = 1$, then find the solution.

Sol. $a_r = a_{r-1} + a_{r-2}$

then corresponding characteristic equation,

$$\alpha^2 - \alpha - 1 = 0$$

Then its solution

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \text{ and } \alpha_2 = \frac{1-\sqrt{5}}{2}$$

Then the solution will be

$$\alpha_r = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r$$

where A_1 and A_2 =constant

$a_0 = 0$ and $a_1 = 1$ (given)

$r = 0 \quad 0 = A_1 + A_2$ ...(1)

$r = 1 \quad 1 = A_1\left(\frac{1+\sqrt{5}}{2}\right)^r + A_2\left(\frac{1-\sqrt{5}}{2}\right)^r$ ...(2)

By solving equation (1) and (2),

$$A_1 = \frac{1}{\sqrt{5}} \text{ and } A_2 = -\frac{1}{\sqrt{5}}$$

$$a_r = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^r - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^r$$

Eg 2. Solve the recurrence relation

$$a_r = 6a_{r-1} - 11a_{r-2} + 6a_{r-3}$$

where $a_0 = 1$, $a_1 = 2$ **and** $a_3 = 6$

Sol. $a_r = 6a_{r-1} - 11a_{r-2} + 6a_{r-3}$

Then corresponding characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

Solving the equation,

$$\alpha = 1, 2, 3$$

So solution is

$$a_r = C_1 + C_2 2^r + C_3 3^r$$

Now if $a_0 = 1$ (given)

$$C_1 + C_2 = C_3 = 1 \quad ...(1)$$

if $a_1 = 2$

$$C_1 + 2C_2 + 3C_3 = 2 \quad ...(2)$$

if $a_3 = 6$

$$C_1 + 4C_2 + 9C_3 = 6 \quad ...(3)$$

Solving these equation (1), (2) and (3)

$$C_1 = 1,\ C_2 = -1 \text{ and } C_3 = 1$$

So solution is $a_r = 1 - 2^r + 3^r$

Eg 3. Solve the relation

$$a_r + 9a_{r-1} + 27a_{r-2} + 27a_{r-3} = 0$$

Sol. The corresponding characteristic equation will be

$$\alpha^3 + 9\alpha^2 + 27\alpha + 27 = 0$$

$$\Rightarrow (\alpha + 3)^3 = 0$$

$$\alpha = -3, -3, -3$$

So solution is $\alpha_r = \left(C_1 r^2 + C_2 r + C_3\right)(-2)^r n$

Generating function :

Let us consider $a_0, a_1, a_2 .... a_r$ is the numeric function of real number, then the function

$$G(z) = a_0 + a_1 z + a_2 z^2 + ... a_r z^r + ...$$

$$= \sum_{n=0}^{\infty} a_n z^n$$

is known as generating function.

Generating function for some numeric functions.

General term of numeric function	Generating function G(z)
$a_r = 1$	$G(z) = \frac{1}{1-z}$
$a_r = r$	$G(z) = \frac{z}{(1-z)^2}$
$a_r = r^2$	$G(z) = \frac{z(z+1)}{(1-z)^3}$

$a_r = r(r+1)$ $\qquad G(z) = \frac{2z}{(1-z)^3}$

$a_r = {}^nC_r$ $\qquad G(z) = (1+z)^n$

Eg 4. Find the generating function for following sequence:

(a) 0,1,-2,4,-8....

(b) 1,0,1,0....

(c) 1,-2,3,-4,5,-6...

(d) 0,1,2,3,4,5,...

Sol. (a) $G(z) = 0 + z - 2z^2 + 4z^3 + ...$

$$= z\left(1-(2z)+(2z)^2\right)$$

$$= \frac{z}{1+2z}$$

(b) $G(z) = 1 + 0(z) + z^2 + 0\left(z^3\right) + z^4 + ...$

$$= 1 + z^2 + z^4 + ...$$

$$= \left(1+z^2\right)^{-1}$$

$$= \frac{1}{\left(1+z^2\right)}$$

(c) $G(z) = 1 - 2z + 3z^2 - 4z^3 + ...$

$$= \frac{1}{1+z^2}$$

(d) $G(z) = 0 + z + 2z^2 + 3z^3 + 4z^4 + ...$

$$= z\left(1 + 2z + 3z^2 + 4z^3 + ...\right)$$

$$= \frac{z}{(1-z)^2}$$

Eg 5. **Find the generating function for following numeric function:**

(a) $a_r = 2^r + 5^r, r \geq 0$

(b) $c(9,r), r \geq 0$

Sol. **(a)** $G(z) = \sum_{r=0}^{\infty} a_r z^r$

$$= \sum_{r=0}^{\infty}\left(2^r + 5^r\right)z^r$$

$$= \sum_{r=0}^{\infty} 2^r z^r + \sum_{r=0}^{\infty} 5^r z^r$$

$$= \frac{1}{1-2z} + \frac{1}{1-3z}$$

$$= \frac{2-5z}{1-5z+6z^2}$$

(b) $G(z) = \sum_{r=0}^{\infty}(9,r)z^r$

$$= c(9,0) + c(9,1)z + c(9,2)z^2 + ...$$

$$= c(9,9)z^9$$

$$= (1+z)^{10}$$

Solution of Linear Recurrence Relation Using Generating Functions:

The solution of linear recurrence relation using generating functions can be understand by following:

Eg: Solve the recurrence relation by the generating function method:

$$a_{n+2} - 3a_{n+1} + 2a_n = 0$$

with initial conditions $a_0 = 2$ **and** $a_1 = 3$

Sol. By the definition of generating function

$$G(a,z) = \sum_{n=0}^{\infty} a_n z^n$$

Now multiply each term of recurrence relation by z^n from $n = 0$ and ∞.

$$\sum_{n=0}^{\infty} a_{n+2} z^n - 3\sum_{n=0}^{\infty} a_{n+1} z^n + 2\sum_{n=0}^{\infty} a_n z^n = 0$$

$$\Rightarrow \left(a_2 + a_3 z + a_4 z^2 + ...\right) - 3\left(a_1 + a_2 z + a_3 z^2 + ...\right) + 2\left(a_0 + a_1 z + a_2 z^2 + ...\right) = 0$$

$$\because G(z) = a_0 + a_1 z + a_2 z^2 + ...$$

$$\Rightarrow \frac{G(z) - a_0 - a_1 z}{z^2} - 3\left[\frac{G(z) - a_0}{z}\right] + 2G(z) = 0$$

Now $a_0 = 2$ and $a_1 = 3$

$$\Rightarrow \frac{G(z) - 2 - 3z}{z^2} - 3\left[\frac{G(z) - 2}{z}\right] + 2G(z) = 0$$

$$\Rightarrow G(z) = \frac{2 - 3z}{1 - 3z + 2z^2} = \frac{2 - 3z}{(1 - z)(1 - 2z)}$$

$$\Rightarrow G(z) = \frac{2 - 3z}{(1 - z)(1 - 2z)} = \frac{A}{(1 - z)} + \frac{B}{(1 - 2z)}$$

$$G(z) = \frac{A(1 - 2z) + B(1 - z)}{(1 - z)(1 - 2z)}$$

So $A = 1$ and $B = 1$

$$G(z) = \frac{1}{1 - z} + \frac{1}{1 - 2z}$$

$$= \sum_{n=0}^{\infty} z_n + \sum_{n=0}^{\infty} (2z)^n$$

$$= \sum_{n=0}^{\infty} \left(1 + 2^n\right) z_n$$

So, required solution is $a^n = 1 + 2^n$

Questions

Very Short type Questions

1. Define the generating functions.
2. Define the recurrence relation.
3. What is linear recurrence relation.

Short Type Questions

1. Solve the recurrence relation.

   $a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$

2. Solve the recurrence relation

   $a_r = a_{r-1} + 2a_{r-2}$

   with $a_0 = 3$ and $a_1 = 10$

3. Find the generating function for the following:

   (a) 2,4,6,8..

   (b) 2,4,8,16...

   (c) 3,-3,3,-3...

(d) 2,0,2,0...

Long type Questions

1. Use generatieng function to solve the recurrence relation:

   $a_r - 7a_{r-1} + 10a_{r-2} = 0 \qquad r \geq 2$

   where $a_0 = 10$ and $a_1 = 41$

2. Solve the recurrence relation

   $a_r = 6a_{r-1} - 11a_{r-2} + 6a_{r-3}$

3. Solve the following recurrence relation

   (i) $a_r + 5a_{r-1} + 6a_{r-2} = 2r^2 - 3r + 1$

   (ii) $a_r + 6a_{r-1} + 12a_{r-2} + 8a_{r-3} = 0$