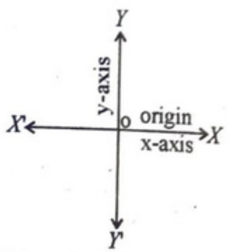
# Chapter 4 Co-Ordinate Geometry

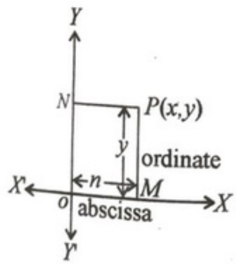
## 4.1 Introduction

Geometry is a very ancient branch of Mathematics. In the year 1637 the famous mathematician René-Des-Cartes studied analytical geometry. In this geometry, position of a point is represented by special numbers called coordinates and the various shapes such as lines, curves etc. are represented in the form of algebraic equations. Hence this branch of Mathematics has been developed by the combination of geometry and algebra. Due to use of coordinates it is also known as co-ordinate geometry.

## 4.2 Cartesian Co-ordinates

We draw two horizontal and vertical lines intersecting each other at right angles at a point O named X'OX and YOY'. There are called coordinate axes. Line X'OX is called x-axis and YOY' is called y-axis and their intersection point O is called origin.




Let P be any point in the plane. Draw PM and PN perpendiculars to x-axis and y-axis respectively. Distance OM and ON are denoted by x and y respectively and are called x-coordinate (abscissa) and y-coordinate (ordinate). The ordered pair (x,y) represents the coordinates of point P.



## 4.3 Signs of Coordinates and quadrants

The coordinate axes X'OX and YOY' divide the coordinate plane into four parts known as quadrants. The quadrants XOY, YOX', X'OY' and Y'OX are respectively called first, second, third and fourth quadrants.

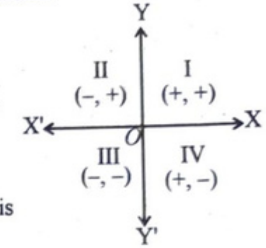
If coordinates of a point are (x, y) and point

(i) lies in the first quadrant then $x > 0, y > 0$

(ii) lies in the second quadrant then $x < 0, y > 0$.

(iii) lies in the third quadrant then $x < 0, y < 0$,

(iv) lies in the fourth quadrant then $x > 0, y < 0$.

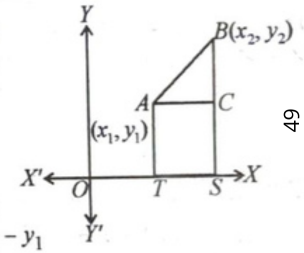



**Note :** (i) If $y = 0$ then the point lies on the x-axis

(ii) If $x = 0$ then the point lies on the y-axis.

## 4.4 Distance between two points

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be two points in the plane. Draw AT and BS perpendicular to x-axis. Now draw AC perpendicular from A to BS.




$\therefore$

$$OT = x_1,\ AT = y_1$$

$$OS = x_2,\ BS = y_2$$

$$AC = TS = OS - OT = x_2 - x_1$$

$$BC = BS - CS = BS - AT = y_2 - y_1$$

$\therefore$ In right triangle $ACB$

$$AB = \sqrt{AC^2 + BC^2}$$

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

**Special case :** Distance of a point $A(x, y)$ from origin (0, 0)

$$OA = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$$

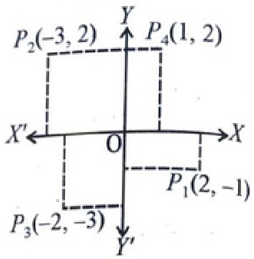
**Note :** Since distance is always positive therefore we take only positive sign of the square root.

$\cos x - \cos y = -2\sin\frac{x+y}{2} \cdot \sin\frac{x-y}{2}$

## ILLUSTRATIVE EXAMPLES

**Example 1. Represent the following points on the paper**

**(i) (2, –1)** **(ii) (–3, 2)** **(iii) (–2, –3)** **(iv) (1, 2)**




**Solution :** (i) Measure two units of length on the right of O to y'. From these points draw lines parallel to the y-axis & x-axis respectively. Their point of intersection $P_1$ is the required point (see the figure)

(ii) Adopting the above procedure, the point $P_2$ is obtained as shown in figure.

(iii) The same process gives the point $P_3$ as shown in figure.

(iv) The same process gives the point $P_4$ as shown in the figure.

**Example 2. Find the distance between the following pair of points.**

**(i) $P(-3, 2)$ and $Q(2, -1)$**

**(ii) $P(a\cos\alpha, a\sin\alpha)$ and $Q(a\cos\beta, a\sin\beta)$**

**(iii) $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$**

**Solution :** (i) Here $x_1 = -3, y_1 = 2, x_2 = 2, y_2 = -1$

Hence $PQ = \sqrt{\{2-(-3)\}^2+(-1-2)^2} = \sqrt{25+9} = \sqrt{34}$

(ii) Here, $x_1 = a\cos\alpha, y_1 = a\sin\alpha, x_2 = a\cos\beta, y_2 = a\sin\beta$

Hence $PQ = \sqrt{(a\cos\beta - a\cos\alpha)^2 + (a\sin\beta - a\sin\alpha)^2}$

$$= a\sqrt{(\cos\beta-\cos\alpha)^2+(\sin\beta-\sin\alpha)^2}$$

$$= a\sqrt{\left[2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}\right]^2+\left[2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}\right]^2}$$

$$= 2a\sin\frac{\alpha-\beta}{2}\sqrt{\left[\sin^2\frac{\alpha+\beta}{2}+\cos^2\frac{\alpha+\beta}{2}\right]}$$

$$= 2a\sin\frac{\alpha-\beta}{2}$$

(iii) Here $x_1 = at_1^2, y_1 = 2at_1, x_2 = at_2^2, y_2 = 2at_2$



$\sin x - \sin y = 2\cos\frac{x+y}{2} \cdot \sin\frac{x-y}{2}$

Hence, $PQ = \sqrt{\left[at_2^2 - at_1^2\right]^2 + \left[2at_2 - 2at_1\right]^2}$

$$= a\sqrt{\left(t_2^2-t_1^2\right)^2+4(t_2-t_1)^2}$$

$$= a\sqrt{(t_2-t_1)^2(t_2+t_1)^2+4(t_2-t_1)^2}$$

$$= a(t_2-t_1)\sqrt{t_2^2+t_1^2+2t_1t_2+4}$$

**Example 3. If the distance between two points $P(-3, 5)$ and $Q(-x, -2)$ is $\sqrt{58}$ then find the value of $x$.**

**Solution :** From the given values

$\sqrt{58} = \sqrt{(-x+3)^2+(-2-5)^2}$ or $58 = (x-3)^2+49$

or $9 = (x-3)^2$ or $x - 3 = \pm 3$

**Example 4. Prove that the points (3, 0), (6, 4) and (–1, 3) are the vertices of a right angled triangle.**

**Solution :** Let $A(3, 0)$, $B(6, 4)$ and $C(-1, 3)$ be the vertices of a triangle $ABC$.

then $AB = \sqrt{(6-3)^2+(4-0)^2} = \sqrt{9+16} = 5$

different

$BC = \sqrt{(-1-6)^2+(3-4)^2} = \sqrt{49+1} = \sqrt{50}$

and, $CA = \sqrt{(3+1)^2+(0-3)^2} = \sqrt{16+9} = 5$

$\therefore \quad AB^2 + AC^2 = BC^2$

Hence $A$, $B$ and $C$ are the vertices of a right angled triangle.

**Example 5. Prove that points $(a, a)$, $(-a, -a)$ and $\left(-\sqrt{3}a, \sqrt{3}a\right)$ are the vertices of an equilateral triangle.** — समबाहु त्रिभुज

**Solution :** Let the given points be $A(a, a)$, $B(-a, -a)$ and $C\left(-\sqrt{3}a, \sqrt{3}a\right)$

Hence $AB = \sqrt{(-a-a)^2+(-a-a)^2} = \sqrt{4a^2+4a^2} = 2\sqrt{2}a$

$BC = \sqrt{\left(-\sqrt{3}a+a\right)^2+\left(\sqrt{3}a+a\right)^2}$

Note अगर equilateral triangle होता उसके तीनों side same आते हैं.

$$= \sqrt{a^2(1-\sqrt{3})^2 + a^2(\sqrt{3}+1)^2} = 2\sqrt{2}\,a$$

$$AC = \sqrt{a^2(\sqrt{3}+1)^2 + a^2(\sqrt{3}-1)^2}$$

$$= 2\sqrt{2a}$$

Clearly, $AB = BC = AC$ Hence the triangle formed by the given points is a equilateral triangle.

## Exercises 4.1

1. Find the distance between the following pair of joints :
   (i) $(a+b, c+d)$ and $(a-b, -c+d)$ (ii) $(a, 0)$ and $(0, b)$
   (iii) $(a \sin\alpha, -b\cos\alpha)$, $(-a\cos\alpha, b\sin\alpha)$
2. If the distance between the points $A(x, 1)$ and $B(-3, -2)$ be 5, then find the value of $x$.
3. If the point $(x, y)$ is equidistant from the points $(a-b, b-a)$ and $(a+b, a+b)$
   Prove that $\frac{x}{a} + \frac{y}{b} = 2$
4. If $A(0, 0)$ and $B(3, \sqrt{3})$ be two vertices of an equilateral triangle, find the third vertex of the triangle.
5. Prove that the points (3, 0), (6, 4) & (–1, 3) are the vertices of a right angled triangle.
6. Prove that the points $A(1, -2)$, $B(3, 6)$, $C(5, 10)$ and $D(3, 2)$ are the vertices of a parallelogram.
7. Prove that the points (2, –2), (6, 4), (5, 7) and (–1, 1) are the vertices of a rectangle.

## Answers 4.1

1. (i) $2\sqrt{(b^2+c^2)}$
   (ii) $\sqrt{a^2+b^2}$
   (iii) $\sqrt{(a^2+b^2)}\,(\sin\alpha + \cos\alpha)$
2. 1 or –7
4. $(0, 2\sqrt{3})$ or $(3, -\sqrt{3})$

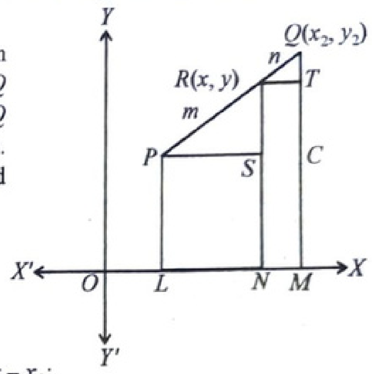


## 4.5 Division of a line segment [R.U. 2016]

A line segment $PQ$ can be divided in a given ratio m : n in two ways:

### (I) Internal Division

When the point of division R lies between $P$ and $Q$ then the division is called internal division




Let $P(x_1, y_1)$ & $Q(x_2, y_2)$ be the given points & let $R(x, y)$ be the point dividing $PQ$ internally in the ratio m : n. From points $P$, $Q$ & $R$ draw lines parallel to $x$-axis & $y$-axis. These lines form similar triangles $RSP$ and $QTR$.

Since the triangle are similar

$$\frac{PR}{RQ} = \frac{PS}{RT} = \frac{RS}{QT} = \frac{m}{n}$$

and
$$PS = LN = ON - OL = x - x_1;$$
$$RT = NM = OM - ON = x_2 - x;$$
$$RS = RN - SN = RN - PL = y - y_1;$$
$$QT = QM - TM = QM - RN = y_2 - y$$

Hence, from $\frac{PS}{RT} = \frac{m}{n}$, $\frac{x-x_1}{x_2-x} = \frac{m}{n}$

or $n(x - x_1) = m(x_2 - x)$ or $x = \frac{mx_2 + nx_1}{m+n}$

$nx - nx_1 - mx_2 + mx = 0$

$x = \frac{mx_2 + nx_1}{m+n}$

Similarly $\frac{RS}{QT} = \frac{m}{n} \Rightarrow \frac{y-y_1}{y_2-y} = \frac{m}{n}$

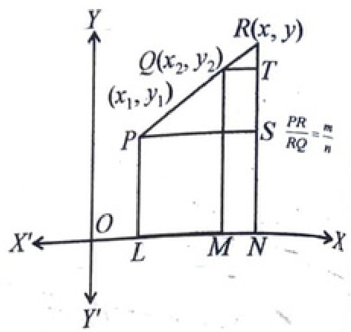
or $n(y - y_1) = m(y_2 - y)$ or $y = \frac{my_2 + ny_1}{m+n}$

$\boxed{y = \frac{my_2 + my_1}{m+n}}$

Hence, The coordinates of $R$ are

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$$

## II. External Division




When the point of division is either on the left of $P$ or on the right of $Q$, such a division is known as external division. The point of division lies on the left of $P$ or on the right of $Q$ according as .

$$\frac{(PR)}{(QR)}<1 \text{ or } \frac{(PR)}{(QR)}>1$$

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the given points. Let $R(x, y)$ be the point dividing $PQ$ externally in the ratio $m : n$. From $P$, $Q$ and $R$ draw lines parallel to $x$ axis & $y$ axis. These lines form similar triangles $PSR$ & $QTR$.

Since the triangles are similar $\frac{PQ}{RQ} = \frac{PS}{QT} = \frac{RS}{RT} = \frac{m}{n}$

and
$$PS = LN = ON - OL = x - x_1;$$
$$QT = MN = ON - OM = x - x_2;$$
$$RS = RN - SN = RN - PL = y - y_1;$$
$$RT = RN - TN = RN - QM = y - y_2;$$

Hence $\frac{PS}{QT} = \frac{m}{n}, \frac{x - x_1}{x - x_2} = \frac{m}{n}$

or $n(x - x_1) = m(x - x_2)$

or $x = \frac{mx_2 - nx_1}{m-n}$

Similarly $\frac{RS}{RT} = \frac{m}{n}, \frac{y - y_1}{y - y_2} = \frac{m}{n}$

or $n(y - y_1) = m(y - y_2)$ or $y = \frac{my_2 - ny_1}{m-n}$

Hence the coordinates of R are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}\right)$$

Note 1 : If R is the middle point of PQ, then R divides PQ internally in the ratio 1 : 1. Hence from the formula of internal division the coordinates of the middle point are



$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

2. If we suppose $\frac{m}{n} = \lambda$, then the point $R$ divides $PQ$ in the ration $\lambda : 1$. Hence the coordinates of R in case of internal division are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$ & in case of external division, the coordinates of R are $\left(\frac{\lambda x_2 - x_1}{\lambda - 1}, \frac{\lambda y_2 - y_1}{\lambda - 1}\right)$

## Illustrative Examples

□ **Example 1 : Find the coordinates of the point dividing the join of $A(1, -2)$ and $B(4, 7)$**

**(i) Internally in the ratio 1 : 2**

**(ii) Externally in the ratio of 2 : 1**

**Solution :** (i) Let $P(x, y)$ be the point of internal division. Here

$$x_1 = 1, y_1 = -2, x_2 = 4, y_2 = 7, m = 1, n = 2$$

$$\therefore \quad x = \frac{1\times4+2\times1}{1+2} = 2 \text{ and } y = \frac{1\times7+2(-2)}{1+2} = 1$$

Hence the required point is (2, 1)

(ii) Let $Q(h, k)$ be the point of external division. Here $m = 2$ and $n = 1$

$$\therefore \quad h = \frac{2\times4-1\times1}{2-1} = 5 \text{ and } k = \frac{2\times7-1\times(-2)}{2-1} = 16$$

IMP Hence the required point is (5, 16)

□ **Example 2 : In what ratio is the line joining $A(8, 9)$ & $B(-7, 4)$ divided by (i) the point (2, 7) (ii) the $x$-axis (iii) the $y$-axis**

**Solution :** (i) Let the point $C(2,7)$ divide the line segment $AB$ in the ratio $\lambda : 1$.

$\therefore$ The coordinates of $C$ are $\left(\frac{-7\times\lambda+8\times1}{\lambda+1}, \frac{4\lambda+9}{\lambda+1}\right)$

But according to the problem the coordinates of $C$ are (2, 7)

Hence, $\quad 2 = \frac{-7\lambda+8}{\lambda+1} \Rightarrow \lambda = \frac{2}{3}$

$\therefore$ Hence, the required ratio is $\frac{2}{3} : 1$ or $2 : 3$.

(ii) The ordinate of any point on the x- axis is zero. Let the segment AB be divided in the ratio $\lambda : 1$ by the x-axis. Hence, the ordinate $y$ of the point of division will be equal to zero.

$$\therefore \quad y = \frac{4\lambda+9}{\lambda+1} = 0 \Rightarrow \lambda = -\frac{9}{4}$$

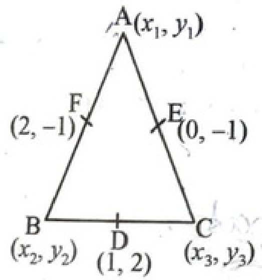
Hence, the x-axis divides the given segment externally in the ratio 9 : 4.

(iii) The abscissa a of any point lying on the $y$-axis is zero. Let the $y$-axis divide the segment in the ratio $\lambda : 1$. Hence the x coordinate of the point of division will be equal to zero.

$$\therefore \quad x = \frac{-7\lambda+8}{\lambda+1} = 0 \text{ or } \lambda = \frac{8}{7}$$

Hence, the $y$-axis divides the given segment internally in the ratio 8 : 7.

**□ Example 3 : If the coordinates of the middle points of the sides of triangle are (1, 2), (0, –1) and (2, –1) respectively, find the coordinates of the vertices of the triangle.**




**Solution :** Let the coordinates of the vertices of the triangle ABC be $A(x_1, y_1)$, $B(x_2, y_2)$ & $C(x_3, y_3)$.

Let the middle points of the sides $BC$, $CA$ & $AB$ be $D(1, 2)$, $E(0, -1)$ and $F(2, -1)$ respectively.

$\because$ D is the middle point of $BC$

$$\therefore \quad \frac{x_2+x_3}{2} = 1 \text{ and } \frac{y_2+y_3}{2} = 2$$

$$\text{or} \quad x_2 + x_3 = 2 \text{ and } y_2 + y_3 = 4 \qquad .....(1)$$

Similarly $E$ & $F$ are the middle points of $CA$ & $AB$

$$\therefore \quad \frac{x_1+x_3}{2} = 0 \text{ and } \frac{y_1+y_3}{2} = 1$$

$$\Rightarrow \quad x_1 + x_3 = 0 \text{ and } y_1 + y_3 = 2 \qquad .....(2)$$

$$\text{and} \quad \frac{x_1+x_2}{2} = 2 \text{ and } \frac{y_1+y_2}{2} = -1$$

$$\Rightarrow \quad x_1 + x_2 = 4 \text{ and } y_1 + y_2 = -2 \qquad ....(3)$$



From (1), (2) and (3)

$$(x_1 + x_2) + (x_2 + x_3) + (x_3 + x_1) = 4 + 2 + 0$$

$$x_1 + x_2 + x_3 = 3 \qquad ....(4)$$

$$\text{and} \quad (y_1 + y_2) + (y_2 + y_3) + (y_3 + y_1) = -2 + 4 - 2$$

$$y_1 + y_2 + y_3 = 0 \qquad ....(5)$$

Similarly from (1), (4), (5)

$$x_1 = 1 \text{ and } y_1 = -4 \;\therefore A(1, -4)$$

Similarly from (2, (4), (5)

$$x_2 = 3 \text{ and } y_2 = 5 \;\therefore B(3, 2)$$

from (3, (4), (5)

$$x_3 = -1 \text{ and } y_3 = 2 \;\therefore C(-1, 2)$$

Hence, the coordinates of the vertices of the triangle ABC are $A(1, -4)$, $B(3, 2)$ and $C(-1, 2)$.

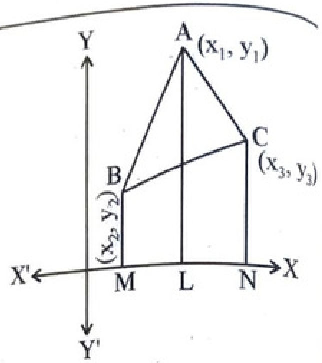
## EXERCISES 4.2

1. Find the coordinates of points of trisection of the line segment joining the points (1, –2) & (–3, 4).
2. In what ratio does the point (4, 4) divide the line segment joining the points (–1, –1) & (7, 7) ?
3. Find the coordinates of the point B such that the point $C(-1, 2)$, divides the line segment joining $A(2, 5)$ & $B$ externally in the ratio 3 : 4.
4. In what ratio does the line $x + y = 4$ divide the line segment joining (–1, 1) & (5, 7).
5. Prove that the point $A(-2, -1)$, $B(1, 0)$, $C(4, 3)$ & $D(1, 2)$ are the vertices of a parallelogram. Is it also a rectangle ?
6. The coordinates of three vertices of a parallogram, in order are (–1, 3), (3, 1) & (2, 2). Find the coordinates of the fourth vertex.
7. Prove that the points $A(4, -1)$, $B(6, 0)$, $C(7, 2)$ & $D(5, 1)$ are the vertices of a rhombus. Is it a square ?

## ANSWERS 4.2

1. $\left(-\frac{1}{3}, 0\right), \left(\frac{-5}{3}, 2\right)$ 2. 5 : 3 3. (3, 6)
4. 1 : 2 6. (–2, 1)

## 4.6. Area of a triangle




Let ABC be a triangle. Let the coordinates of its vertices be $(x_1, y_1)$, $(x_2, y_2)$ & $(x_3, y_3)$ respectively. From $A$, $B$ & $C$ draw perpendiculars $AL$, $BM$ & $CN$ on the x-axis. Thus we get the trapeziums

ABML, CALN & CBMN.

$\therefore$ Area of $\Delta$ABC = Area of the trapezium ABML + Area of the trapezium CALN – Area of the trapezium CBMN

$\therefore$ Area of $\Delta ABC$ $\Delta = \frac{1}{2}ML(BM+AL) + \frac{1}{2}LN(AL+CN) - \frac{1}{2}MN(BM+CN)$

$$= \frac{1}{2}(x_1 - x_2)(y_2 + y_1) + \frac{1}{2}(x_3 - x_1)(y_1 + y_3) - \frac{1}{2}(x_3 - x_2)(y_2 + y_3)$$

$$= \frac{1}{2}[(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)]$$

$$= \frac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]$$

$$= \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ (in determinant form)}$$

**Note 1 :** While calculating the area of $\Delta$ABC of A, B & C are taken in an anticlockwise order then the area of considered positive. If A, B, C are taken in clockwise order then the area comes out to be negative. But the value of the area is always taken to be positive.

2. To find the area of a polygon, we divide it into a number of triangles. The sum of the magnitude of the area of these triangles gives the area of the polygon.

3. Three points $A(x_1, y_1)$, $B(x_2, y_2)$ & $C(x_3; y_3)$ are collinear iff (i) area of $\Delta$ABC = 0

$$\text{i.e. } \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ or}$$

(ii) $AB + BC = AC$ or $AC + BC = AB$ or $AC + AB = BC$



## Illustrative Examples

**Example 1 : Find the area of the triangle whose vertices are $A(1, 1)$, $B(7, -3)$ & $C(12, 2)$**

**Solution :** By the formulae for the area of the triangle

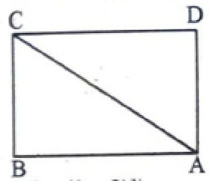
$$\Delta = \frac{1}{2}|[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_3)]|$$

$$\Delta = \frac{1}{2}|[1(-3-2) + 7(2-1) + 12(1+3)]|$$

$$= \frac{1}{2}|[-5 + 7 + 48]|$$

$$= 25 \text{ sq. unit}$$

**Example 2 : Find the area of quaderilateral whose vertices are $A(1, 1)$, $B(7, -3)$, $C(12, 2)$ & $D(7, 21)$.**




**Solution :** Area of the quaderilateral $ABCD$ = |Area of $\Delta ABC$| + |Area of $\Delta ACD$|

Now area of $\Delta ABC$ = 25 sq. unit (by eg.1)

and area of $\Delta ABC = \frac{1}{2}|[1 + (2-21) + 12 \times (21 - 1 + 7 \times (1-2)]|$

$$= \frac{1}{2}|(-19 + 240 - 7)| = 107 \text{ sq.unit}$$

$\therefore$ Area of the aquacultural $ABCD$ = 25 + 107 = 132 sq. unit.

**Example 3 : Prove that the points $A(a, b+c)$, $B(b, c+a)$ and $C(c, a+b)$ are collinear.**

**Solution :** By determinant method, area of the triangle

$$= \frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix} \text{ applying } C_2 + C_1$$

$$= \frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix}$$

$$= \frac{1}{2}(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix} = 0 \qquad [\because C_2 = C_3]$$

Hence, the three points are collinear.

**Example 4 : Find the value of k so that the points $(k, 2-2k)$, $(-k+1, 2k)$ & $(-4-k, 6-2k)$ may be collinear.**

**Solution :** Let the three given points be

$$A(x_1, y_1) \equiv A(k, 2-2k)$$
$$B(x_2, y_2) \equiv B(-k+1, 2k)$$

and
$$C(x_3, y_3) \equiv C(-4-k, 6-2k)$$

If the points are collinear then

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 0$$
$$\Rightarrow k(2k - 6 + 2k) + (-k + 1)(6 - 2k - 2 + 2k) + (-4 - k)(2 - 2k - 2k) = 0$$
$$\Rightarrow k(4k - 6) - 4(k - 1) + (4 + k)(4k - 2) = 0$$
$$\Rightarrow 4k - 6k - 4k + 4 + 4k2 + 14k - 8 = 0$$
$$\Rightarrow 8k^2 + 4k - 4 = 0$$
$$\Rightarrow (2k - 1)(k + 1) = 0$$

$\Rightarrow k = \frac{1}{2}$ or $k = -1$, Hence, for the given points to be collinear

$$k = \frac{1}{2} \text{ or } k = -1$$

## EXERCISES 4.3

1. Find the area of the triangle whose vertices are (1, 3), (2, 4) & (5, 6).
2. Find the area of the equilateral whose vertices are $A(3, 4)$, $B(-4, 0)$, $C(5, -1)$ & $D(6, 2)$.
3. Prove that the points (–1, –1), (2, 3) & (8, 11) are collinear.
4. Find the condition of colinearity of the points $(a, 0)$, $(0, b)$ & (1, 1).
5. The vertices of the $\Delta ABC$ are (4, 2), (4, 5) & (–2, 2) respectively D, E & F are the middle points of the sides $BC$, $CA$ & $AB$ respectively. Prove that Area of $\Delta ABC = 4 \times$ area of $\Delta DEF$.
6. Find the value of $x$ so that the points $(x, -1)$, (2, 1) & (4, 5) are collinear.

## ANSWERS 4.3

1. $\frac{1}{2}$ sq unit 2. 27 sq. unit

3. $\frac{1}{a} + \frac{1}{b} = 1$ 6. 1

# Chapter 5 Locus and Straight Line

## 5.1 Locus

[R.U. 2015]

**Definition :** When a point moves under a given condition then the path traced out by the point is called the locus of the point. Thus the locus of a point is a curve such that the points of the curve satisfy the given conditions.

**Example :** A circle is the locus of a point which moves such that its distance from a given fixed point is always constant.

## 5.2 Equation of Locus

The equation of locus of a point is an algebraic relation between the coordinates x and y of a general point $P(x, y)$. This relationship is such that it is satisfied by the coordinates of all points of the locus.

## 5.3 Method of Finding the Locus

When the conditions on the movement of the point are given, we adopt the following procedure to find its locus :

(i) We choose some arbitrary point (h, k) on the locus.

(ii) We establish a relation between h and k on the basis of the given condition or conditions.

(iii) The algebraic relation is simplified if it is needed.

(iv) Finally, we change to current coordinates by replacing h by x and k by y.

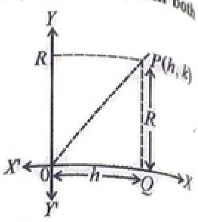
The equation so obtained is the required locus.

**Note :** If the equations of two curves are given in parametric form, the locus of their point of intersection is obtained by eliminating the "parameter" from these equations.

$(x-h)^2+(y-k)^2=a^2$

## Illustrative Examples

■ **Example 1 : Find the locus of a point which is equidistant from both the axes.**

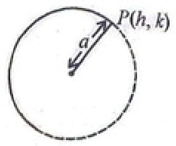



**Solution :** Let $P(h, k)$ be a variable point. According to the given condition the distance of P from x-axis and y-axis are equal.

$\therefore$ $PQ = PR$ or $k = h$ changing the current coordinates, putting $h = x$ and $k = y$, the required locus of the point $P(h, k)$ is $y = x$ which is shown in fig. by the line $OP$.

■ **Example 2 : Find the locus of a point which is always at a constant distance $a$ from the origin.**

**Solution :** Let the $P(h, k)$ be the variable point. According to the problem, the distance of P from $O(0, 0) = a$



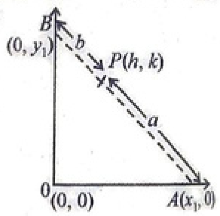

$$\therefore \quad \sqrt{(h-0)^2+(k-0)^2} = a$$

Squaring both side

$$(h-0)^2 + (k-0)^2 = a^2$$
$$h^2 + k^2 = a^2$$

Changing to current coordinates by replacing $h$ by $x$ and $k$ by $y$ we get $x^2 + y^2 = a^2$ which is the required locus.

■ **Example 3 : A line segment $AB$ of length $(a + b)$ moves such that the end $A$ is always on $x$-axis and the end $B$ is always on $y$-axis. $P$ is a point on $AB$ such that $PA = a$ and $PB = b$. Find the locus of $P$.**




**Solution :** Let the ends $A$ and $B$ of the line segment be at a distance $x_1$ and $y_1$ respectively from the origin. Hence, the coordinates of $A$ and $B$ are $(x_1, 0)$ and $(0, y_1)$ respectively.

In right angled triangle $OAB$,

$$OA^2 + OB^2 = AB^2$$

or $$x_1^2 + y_1^2 = AB^2 \qquad ....(1)$$

Let the coordinates of the variable point $P$ be $(h, k)$. $P$ divides $AB$ in the ratio $a : b$.

$$\therefore \quad h = \frac{a\times 0 + b\times x_1}{a+b} \;;\; k = \frac{a\times y_1 + b\times 0}{a+b}$$



$$h = \frac{bx_1}{a+b} \quad ; \quad k = \frac{ay_1}{a+b}$$
$$h(a+b) = bx_1 \quad ; \quad k(a+b) = ay_1$$
$$\therefore \quad x_1 = \frac{h(a+b)}{b} \quad ; \quad y_1 = \frac{k(a+b)}{a}$$

Putting the values of $x_1$ and $y_1$ in eq (1)

$$\left[\frac{h(a+b)}{b}\right]^2 + \left[\frac{k(a+b)}{a}\right]^2 = (a+b)^2$$

or $$\frac{h^2(a+b)^2}{b^2} + \frac{k^2(a+b)^2}{a^2} = (a+b)^2$$

$$(a+b)^2\left[\frac{h^2}{b^2}+\frac{k^2}{a^2}\right] = (a+b)^2$$

Changing to current coordinates i.e. putting

$$h = x \text{ and } k = y$$

$$\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$$

Which is the required locus.

■ **Example 5 : Find the locus of a point whose coordinates are given by $x = at^2$, $y = 2at$, where $t$ is a variable.**

**Solution :** According to the problem $x = at^2$ and $y = 2at$ Here $t$ is a variable and the required locus will be obtained by eliminating $t$.

Hence, $\quad y = 2at$ or $t = \dfrac{y}{2a}$

Putting this value in $x = at^2$ or $x = a\dfrac{y^2}{4a^2}$ or $x = \dfrac{y^2}{4a}$

Hence $y^2 = 4ax$ is the required locus.

$(x, y)$

$(x-4)^2+(y-0)^2=(x-0)^2+(y-3)^2$

$\cancel{x^2}-8x+16+\cancel{y^2}=\cancel{x^2}+\cancel{y^2}-6y+9$

$8x-6y-7=0$

### Exercises 5.1

1. Find the locus of the point of whose
   (i) Distances from the point (4, 0) the point (0, 3) are equal.
   (ii) Distance from $x$-axis is four times its distance from $y$-axis.

$y=4x$ $\quad$ $x=4y$

2. A point moves such that the sum of squares of its distances from two fixed points $(a, 0)$ and $(-a, 0)$ is constant equal to $2c^2$. Find the locus of the point.
3. Find the locus of a point whose distance from the point $(-g, -f)$ is $c$.
4. $A(2, 0)$, $B(-2, 0)$ and $C(3, -3)$ are three given points. Find the locus of a point P such that $PA^2 + PB^2 = 2PC^2$.
5. Find the locus of the point, the sum of whose distances from the points $(ae, 0)$ and $(ae, 0)$ is $2a$, is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $b^2 = a^2(1 - e^2)$.

### Answers 5.1

1. (i) $8x - 6y = 7$ (ii) $y = 4x$
2. $x^2 + y^2 = c^2 - a^2$
3. $x^2 + y^2 + 2gx + 2fy + g^2 + f^2 - c^2 = 0$
4. $3x - 3y = 7$
5. $y^2 = 4ax$

## 5.4 Straight Line

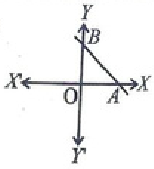
**Definition :** A straight line is the locus of a point having the property that if any two points of the line are joined by the shortest distance then all the points of the locus lie on it.

## 5.5 Equation of Straight Line

The equation of a line is a relation between the abscissa $x$ and the ordinate $y$ of a general point on the line which is satisfied by the coordinates of all the points on the line and by no other point.

## 5.6 Definitions

**(a) Intercept :** If the line $AB$ intersects the $x$-axis at $A$ and $y$-axis at B then.




(i) $OA$ is called the intercept of the line $AB$ on x-axis.

(ii) $OB$ is called the intercept of the line on y-axis.

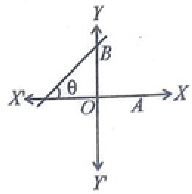
(iii) $OA$ and $OB$ are called the intercepts of the line on the axes.

**Note :** If the line intersects the $x$-axis or the y-axis on the negative side of the origin, then the intercepts are taken as negative.

**(b) Slope of a line :** The slope of a line is the tangent of the angle which the line

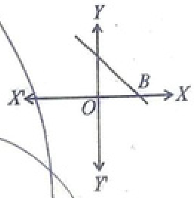


makes with the positive direction of the x-axis.




The slope of a line is generally denoted by m. If the line makes an angle θ with the positive direction (↶) of x-axis, then $m = \tan\theta$. If the line makes an angle θ with the negative direction (↷) of x-axis then $m = -\tan\theta$. If the line coincides with the x-axis, then $\theta = 0$, hence the slope of x axis or a line parallel to x-axis is

$$m = \tan 0 = 0$$

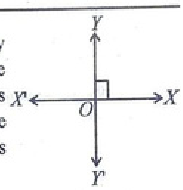



Similarly, if the line coincides with y-axis or is parallel to y-axis then it makes an angle of 90° with the positive direction of x-axis. Hence the slope of y-axis or a line parallel to y ais is $m = \tan 90° = \infty$.

If the line makes equal angles with the axes then it makes an angle of 45° with the positive direction of x-axis. Hence its slope $m = \tan 45° = 1$.

**Note :** The angle made by a line with the positive direction of x-axis (measured anticlockwise) always lies between 0° and 180°.

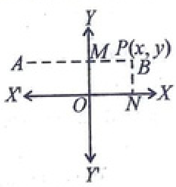
## 5.7 Rectangular axes




If the lines representing x-axis and y-axis are mutually perpendicular, then they are called rectangular axes. The ordinate or the $y$ coordinate of any point lying on the $x$-axis is always zero. Hence, the equation of the x-axis is $y = 0$. The abscissa or the $x$ coordinate of any point lying on the y-axis is always zero Hence, the equation of the y-axis is $x = 0$.

## 5.8 Equation of line parallel to axes

**(i) Equation of a line parallel to x-axis and at a distance of b from it:**




Let $AB$ be a line parallel to $x$-axis and at a distance of from it. Let $AB$ interest the y-axis at a point m, above the origin.

Let $P(x, y)$ be any point on the line $AB$. From $P$ draw $PN$ perpendicular to x-axis. For the point $P$, ordinate $PN = y$ but $PN = OM$.

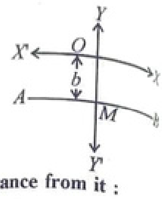
Hence $OM = y$

∴ $y = b$

Thus the ordinate of every point on the line $AB$ is equal to b.

$\therefore$ **The equation of the line $AB$ is $y = b$.**

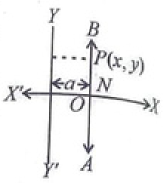
**Note :**



(a) If the line $AB$ is below the x-axis at a distance $b$, then its equation is $y = -b$.

(b) If the line $AB$ coincides with the x-axis, then $b = 0$. Then the equation of the line is same as the equation of the x-axis i.e. $y = 0$.

**(ii) Equation of a line parallel to y-axis and at a distance from it :**



Let $AB$ be a line parallel to y-axis and at a distance '$a$' from it. Let $AB$ intersect the x-axis on the positive side at the point N.

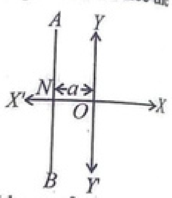
Let $P(x, y)$ be any point on the line from P. Draw $PM$ perpendicular on the y axis, i.e. for the point P, abscissa $PM = x$. But $PM = ON$

$\therefore \quad ON = x$

$\therefore \quad x = a$

Similarly for each point on the line $AB$ the abscissa a $x$ is equal to a. Hence the equation of the line $AB$ is $x = a$.

**Note :**



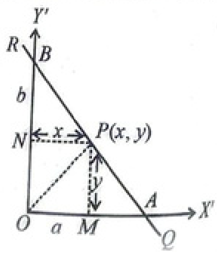
(a) If the line $AB$ intersects on the negative side of $y$-axis, i.e. to the left of y-axis then its equation is $x = -a$.

(b) If the line $AB$ coincides with the y-axis, then $a = 0$. Hence, the equation of the line $AB$, i.e. the y-axis will be $x = 0$.

## 5.9 Equations of straight lines in various standard forms [R.U. 2016]

### (A) Intercept form

**Equation of a line cutting off intercepts a and b on the x-axis and y-axis respectively.**



Let the line $QR$ intersect the x-axis and y-axis at the point $A$ and $B$ respectively such that $OA = a$ and $OB = b$. Let $P(x, y)$ be any point on the line. Join $OP$ draw perpendiculars $PM$ and $PN$ from $P$ on the $x$ and $y$-axis respectively.



Area of right $\Delta OAB$ = area of $\Delta OPA$ + Area of $\Delta OPB$.

$$\frac{1}{2} \times OA \times OB = \frac{1}{2} \times OA \times PM + \frac{1}{2} + OB \times PN$$

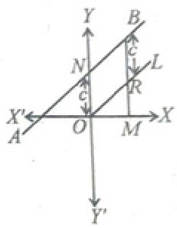
$$\frac{1}{2} \times a \times b = \frac{1}{2} \times a \times y + \frac{1}{2} \times b \times x$$

Dividing each term by $\frac{1}{2} \times a \times b$, we get $\boxed{\frac{x}{a} + \frac{y}{b} = 1}$

Hence the required equation of the line is $\frac{x}{a} + \frac{y}{b} = 1$, which is called the intercept form of the equation of the line.

### (B) Slope Intercept form

**Equation of a line cutting intercept c on the y-axis making an angle θ with the positive direction (↶) of the x-axis**



Let $P(x, y)$ be any point on the line $AB$. from $P$, draw $PM$ perpendicular on $OX$ which intersects the line $OL$ through the origin $O$ and parallel to $AB$ at $R$.

In right $\Delta OMR$, $\tan\theta = \frac{RM}{OM}$,

or $\quad RM = OM \tan\theta$

Now, $\quad PM = PR + RM \ (AB \| OL)$

$PM = C + OM \tan\theta \ (ON = RP = c)$ $\qquad ax+by+c=0$

$y = x \tan\theta + c$

or $\quad \boxed{y = mx + c}$ $\qquad m = -\frac{b}{a}$

Here $m = \tan\theta$, $m$ = slope of the line.

This is the required equation of the line.

When θ is an obtuse angle, then the slope $m$ is negative and when θ is an acute angle the value m is positive. If the line makes equal angles with axes, then $m = \pm 1$.

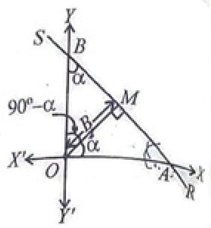
### (C) Normal Form

**Equation of the line when the length of the perpendicular from the origin and the angle made by this perpendicular with the $x$-axis is given.**

Let the line $RS$ intersect the x-axis and the y-axis at $A$ and $B$ respectively. $OM$ is the perpendicular on the line from the origin. Let $OM = p$ and this perpendicular makes an angle α with the positive direction of x-axis i.e. $\angle MOA = \alpha$.

In right $\Delta OBM$

$$\begin{aligned}\angle OBM &= 90^\circ - \angle BOM\\ &= 90^\circ - [90^\circ \angle MOA]\\ &= 90^\circ - [90^\circ - \alpha]\\ &= \alpha\end{aligned}$$




In right $\Delta OMA$ $\cos\alpha = \frac{p}{OA} \Rightarrow OA = \frac{p}{\cos\alpha}$

in right $\Delta OBM$ $\sin\alpha = \frac{p}{OB} \Rightarrow OB = \frac{p}{\sin\alpha}$

Hence, the line $RS$ cuts intercepts $OA$ and $OB$ i.e. $\frac{p}{\cos\alpha}$ and $\frac{p}{\sin\alpha}$ on the x-axis and y-axis respectively.

Hence its equation is $\boxed{\frac{x}{p/\cos\alpha} + \frac{y}{p/\sin\alpha} = 1}$

or $x\cos\alpha + y\sin\alpha = p$

Which is the required equations of the line.

**Note :** (i) The length of the perpendicular p, drawn from the origin on the line is always taken as positive.

(ii) The angle made by this perpendicular with the positive direction of x-axis may lie between 0° and 360°.

## Illustrative Examples

**Example 1 : Find the equation of the line parallel to the y-axis and passing through the point (4, 3).**

**Solution :** The equation of a line parallel to y-axis is

$$x = a \qquad .....(1)$$

The line passes through (4, 3). Hence the coordinates satisfy the equation of the line.

$\therefore$ $4 = a$ putting the value of a in equation (1), the equation of the required line is $x = 4$.

**Example 2 : Find the equation of the line which is equidistant from the lines $y = 8$ and $y = -14$.**

**Solution :** We known that the lines $y = 8$ and $y = -14$ are both parallel to x-axis Hence a line equidistant from them will also be parallel to x-axis whose distance from x-axis is $\frac{8+(-14)}{2} = -3$

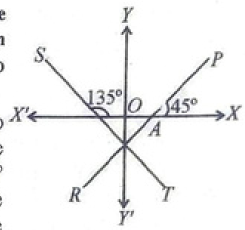


Hence the equation of the required line is $y = -3$

**Example 3 : Find the equation of the line which cuts an intercept of 5 units from the negative side of the y-axis and makes an angle $\tan^{-1}\sqrt{3}$ with the x-axis.**

**Solution :** Let the equation of the line be $y = mx + C$. According to the problem, the intercept on the y axis $C = -5$ units. The angle made with the x-axis $\theta = \tan^{-1}\sqrt{3}$ $\therefore$ $\tan\theta = \sqrt{3}$. Putting values of $m$ and $C$ in the equation, the equation of the line is $\sqrt{3}x - y - 5 = 0$.

**Example 4 : Find the equation of the line which cuts an intercept of 3 units below the origin on the y-axis and is equally inclined to the two axis.**




**Solution :** The required line is equally inclined to the two axes. We know that the angle between the axes is 90°. Hence, the line will make an angle of 45° or 135° with the positive direction of x-axis. Let the equation of the line in the two situations given in the problem be

$$y = mx + c. \qquad .....(1)$$

Here $c = -3$ and $m = \tan 45^\circ \tan 135^\circ$

$\Rightarrow$ $m = 1$ or $-1$

Now putting the values of $m$ and $c$ in equation (1)

$$y = x - 3$$

or $y = -x - 3$

$\Rightarrow$ $x - y - 3 = 0$

or $x + y + 3 = 0$

Hence the equation of the required line is $x - y - 3 = 0$ or $x + y + 3 = 0$.

**Example 5 : Find the equation of the line passing through the point (2, 3) and cuts intercepts on the axis which are of equal magnitude but of opposite sign.**

**Solution :** Let the equation of the required line be

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad .....(1)$$

Where $a$ and $b$ are the lengths of intercepts on x and y-axis respectively. Now, according to the problem $b = -a$. Hence, the equation of the line is

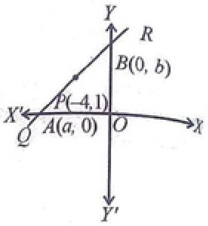
$$\frac{x}{a}+\frac{y}{-a}=1$$

or $$x-y=a \quad ...(2)$$

This line passes through (2, 3). Hence the coordinates satisfy the equation of the line. Putting $x = 2$, $y = 3$ in equation (2), we have $a = -1$. Putting the values of a in equation (2), we get

$$x-y=-1 \text{ or } x-y+1=0$$

which is the required equation.




**◻ Example 6 : Find the equation of the line passing through the point (–4, 1) and this point divides the intercept of the line between the axes in the ratio 1 : 2.**

**Solution :** Let the equation of the line be $\frac{x}{a}+\frac{y}{b}=1$ where a and b are the intercepts on the axes.

The line intersects the axes at $A$ and $B$ whose coordinates are $(a, 0)$ and $(0, b)$. If $P(x_1, y_1)$ be the point dividing $AB$ in ratio 1 : 2, then

$$x_1=\frac{2\times a+1\times 0}{2+1}, \quad y_1=\frac{2\times 0+1\times b}{2+1}$$

or $$x_1=\frac{2a}{3}, \; y_1=\frac{b}{3}$$

According to the problem this point is (–4, 1)

$$\therefore \quad \frac{2a}{3}=-1 \text{ and } \frac{b}{3}=1$$

$$a=-6 \text{ and } b=3$$

Putting the values of a and b in equation (1), we have

$$\frac{x}{-6}+\frac{y}{3}=1$$

Hence, the equation of the required line in

$$x-2y+6=0$$

**◻ Example 7 : Find the equation of the line such that the length of the perpendicular drawn from the origin to the line is 5 units and this perpendicular makes an angle 135° with $x$-axis.**

**Solution :** Let the equation of the line be

$$x\cos\alpha+y\sin\alpha=p \quad ....(1)$$



Here $\alpha = 135°$ and $p = 5$ units.

$$\cos\alpha=\cos 135°=-\frac{1}{\sqrt{2}} \text{ and } \sin\alpha=\sin 135°=\frac{1}{\sqrt{2}}$$

Putting the values in equation (1),

$$x\left(-\frac{1}{\sqrt{2}}\right)+y\left(\frac{1}{\sqrt{2}}\right)=5$$

$$-\frac{x}{\sqrt{2}}+\frac{y}{\sqrt{2}}=5 \text{ or } x-y+5\sqrt{2}=0$$

Hence the equation of the required line $x-y+5\sqrt{2}=0$.

**◼ Example 8 : Find the equation of line, passing through point (–2, 8) and forms 45° angle with X-axis ?** **[R.U. 2015]**

**Solution :** $m=\tan 45°=1$

$\therefore$ Equation of required line, using one point form

$$y-8=1(x+2)$$

$$\Rightarrow \quad y-8=x+2$$

$$\Rightarrow \quad x-y+10=0$$

## EXERCISES 5.2

1. Find the equation of the line parallel to the x-axis and
   (i) Is at a distance of 5 units above the origin
   (ii) Is at a distance of 3 units above the origin.
2. Find the equations of the lines parallel to the x-axis and situated at a distance (i) $a+b$ (ii) $a^2-b^2$ (iii) $b\cos\theta$ from it.
3. Find the equations of the lines parallel to y-a axis at at distance (i) 5 (ii) – 3 (iii) 2/5 units from the origin.
4. Find the equations of the lines parallel to y-axis and situated at distances (i) $\sqrt{7}$ (ii) $-\sqrt{3}+2$ (iii) p + q from it.
5. Find the equations of the lines passing through (–3, 2) and (i) perpendicular to x axis (ii) parallel to x -axis.
6. Find the equations of the lines passing through the point (3, 4) and parallel to the $x$ and $y$ axes. Also find the equations of lines parallel to these lines and at a distance of 8 units from these lines.
7. Find the points of intersection of the lines $x = \pm 4$ and $y = \pm 3$. Also find the area of the rectangle formed by these lines.
8. Find the equations of the lines passing through the origin

(i) Making an angle of $-135°$ with x-axis.

(ii) Making an angle of $60°$ with OY in first quadrant.

9. Find the equations of the lines whose intercept on x-axis and y-axis are (i) 5, 3 (ii) $-2$, 3 respectively.

10. Find the equation of the line passing through the point (2, 3) and which cuts equal intercepts on the axes.

11. Find the equation of the line passing through the point (1, 2) such that the intercept on x-axis is twice the intercept on $y$-axis.

12. Find the equation of the line passing through the point $(-3, -5)$ such that the intercept of the line between the axis is bisected at this point.

13. Find the equations of two lines passing through the point $(4, -3)$ such that the sum of the intercepts on the axes is 5 units.

14. A line cuts instercepts of length 5 and 3 from the axis. Find the equation of the line when the intercepts.

(i) Are on the positive sides of the axis.

(ii) Are one the negative sides of the axis.

(iii) The first intercept is on the positive side and the other is on the negative side.

15. The perpendicular from the origin on a line makes and angle of $30°$ with y-axis and its length is 2 units. Find the equation of the line.

16. Find the equation of the line such that the length of perpendicular from the origin on the line is 3 units and the perpendicular makes an angle of $\tan^{-1}\left(\frac{5}{12}\right)$ with the positive direction of x-axis.

17. Find the locus of middle points of the intercepts of the line $x \cos \alpha + y \sin \alpha = 3$, between the axes where $\alpha$ is a variable.

18. Find the equation of the line for which $p = 3$ and $\cos \alpha = \frac{\sqrt{3}}{2}$, where p is the length of the perpendicular on the line from the origin and $\alpha$ is the angle which the perpendicular makes with x-axis.

### ANSWERS 5.2

1. (i) $y = 5$ (ii) $y + 3 = 0$
2. (i) $y = a + b$ (ii) $y = a^2 - b^2$ (iii) $y = b \cos \theta$



3. (i) $x = 5$ (ii) $x + 3 = 0$ (iii) $5x - 2 = 0$
4. (i) $x = \sqrt{7}$ (ii) $x = 2 - \sqrt{3}$ (iii) $x = p + q$
5. $x + 3 = 0, y = 2$
6. $y = 4, x = 3, y = 12$ and $y + 4 = 0, x = 11$ and $x + 5 = 0$
7. (4, 3), (–4, 3), (–4, – 3), (4, – 3) Area = 48 se. unit
8. (i) $x - y = 0$ (ii) $x - \sqrt{3}y = 0$ (iii) $x - y + 5 = 0$
9. (i) $3x + 5y - 15 = 0$ (ii) $3x - 2y + 6 = 0$
10. $x + y = 5$ 11. $x + 2y = 5$
12. $5x + 3y + 30 = 0$ 13. $3x + 2y - 6 = 0, x - 2y - 10 = 0$
14. (i) $3x + 5y - 15 = 0$ (ii) $3x + 5y - 39 = 0$ (iii) $3x - 4y - 15 = 0$
15. $x - \sqrt{3}y + 4 = 0$ 16. $12x + 5y - 39 = 0$
17. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{9}$ 18. $\sqrt{3}x + y = 6, \sqrt{3}x - y = 6$

$-\frac{b}{a}$ m=

## 5.10 STRAIGHT LINE AND LINEAR EQUATION IN $x$ AND $y$

(a) Any line in a plane is represented by an equation of first degree in x and y.

The angle made by any line in the plane with x-axis is either an acute angle or a right angle or an obtuse angle. There is no other possibility.

(i) If the line makes an acute angle with x-axis, than its equal is of the form $y = mx + c$ where $m = \tan \theta$.

(ii) If the line make a right angle with x-axis, then the line parallel to y-axis and the equation of the line is of the form $x = c$.

(iii) If the line makes and obtuse angle with $x$-axis, then also its equation is of the form $y = mx + c$ as in case (i).

Hence, in all three cases the equation of the line is a linear equation in x and y.

(b) A linear equation in $x$ and $y$ always represents straight line :

If A, B, C are constant, then the general from of a linear equation in x and y is

$$Ax + By + C = 0$$

Here the highest power of $x$ is one and the minimum is zero. In the first and second terms the powers of $x$ and $y$ are one while the third term is free from $x$ and $y$ where the powers of $x$ and $y$ are zero. Hence equation (i) is the general equation of first degree in $x$ and $y$.

In equation (1) the values of A and B cannot be both zero simultaneously since in that case the equation reduces to $C = 0$ which is not possible for all values of C and in absence of a variable, the equation becomes meaningless.

Hence, in the equation $Ax + By + C = 0$ either $A \neq 0$ or $B \neq 0$.

## 5.11 Reduction of General Equation of Straight Line into Standard Forms

**1. To express in slope intercept form $y = mx + c$ :** We know the general equation of a line is $Ax + By + C = 0$

or
$$By = -Ax - C$$
$$y = \left(-\frac{A}{B}\right)x - \left(\frac{C}{B}\right) \text{ (when } B \neq 0)$$
$$y = \left(-\frac{A}{B}\right)x + \left(\frac{-C}{B}\right)$$

$m = \frac{-A}{B}$
$$y = mx + C \qquad \text{Where } m = -\frac{A}{B},\ c = -\frac{C}{B}$$

**Note :** (i) The slope of the line $Ax + By + C = 0$
$$m = -\frac{A}{B} = -\frac{coeff.of\ x}{coeff.of\ y}$$

(ii) intercept on y-axis $= -\frac{C}{B}$

**2. To express intercept form $\frac{x}{a} + \frac{y}{b} = 1$ :** The general equation of the line is $Ax + By + C = 0$ or $Ax + By = -C$

$$\Rightarrow \quad \frac{x}{\left(-\frac{C}{A}\right)} + \frac{y}{\left(-\frac{C}{B}\right)} = 1 \text{ or } \frac{x}{a} + \frac{y}{b} = 1$$

Where $\quad a = -\frac{C}{A}$ and $b = -\frac{C}{B}$

The intercepts on x-axis and y-axis are $-\frac{C}{A}, -\frac{C}{B}$, respectively.

**3. To express in the normal form $x \cos\alpha + y \sin\alpha = p$ :** The general equation of the line is



$$Ax + By + C = 0 \text{ or } Ax + By = -C \quad \text{....(1)}$$

Let the normal form of the line be

$$x \cos\alpha + y \sin\alpha = p \text{ (Here p is positive)} \quad \text{.....(2)}$$

Equations (1) and (2) represent the same line hence comparing the like terms.

$$-\frac{C}{p} = \frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \pm\frac{\sqrt{A^2+B^2}}{\sqrt{\cos^2\alpha + \sin^2\alpha}}$$

or
$$-\frac{C}{p} = \frac{A}{\cos\alpha} = \frac{B}{\sin\alpha} = \pm\frac{\sqrt{A^2+B^2}}{1}$$

Hence, $p = \frac{-C}{\pm\sqrt{A^2+B^2}}$, $\cos\alpha = \frac{A}{\pm\sqrt{A^2+B^2}}$, $\sin\alpha = \frac{B}{\pm\sqrt{A^2+B^2}}$

on putting the values in equation (2)

$$x\frac{A}{\pm\sqrt{A^2+B^2}} + y\frac{B}{\pm\sqrt{A^2+B^2}} = -\frac{C}{\pm\sqrt{A^2+B^2}}$$

i.e. $$\frac{A}{\pm\sqrt{A^2+B^2}}x + \frac{B}{\pm\sqrt{A^2+B^2}} = -\frac{C}{\pm\sqrt{A^2+B^2}}$$

Hence this $x \cos\alpha + y \sin a = p$ is the required normal form of the line.

Remember that in the equation $p = \frac{C}{\pm\sqrt{A^2+B^2}}$ sign from ± is to be taken which makes the equation positive.

**Note :** (i) To reduce the general equation $Ax + By + C = 0$ to normal form, first of all C is transferred to R.H.S. and its made positive if not so, by dividing both sides by $-1$.

(ii) Each term is divided by $\sqrt{A^2+B^2}$ i.e. by $\sqrt{(coeff.of\ x)^2 + (coeff.of\ y)^2}$

## 5.12 Straight line passing through a point

**To find the equation of the line passing through the point $(x_1, y_1)$ and making an angle θ with x-axis.**

Since the line makes an angle θ with the x-axis, the slope of the line $m = \tan\theta$. Let the equation of the line be

$$y = mx + c$$

The line passes through $(x_1, y_1)$ hence it satisfies equation (1),

$$y_1 = mx_1 + c$$

Subtracting equation (2) from equation (1).

$$y - y_1 = m(x - x_1)$$

Which is the equation of the required line.

**Note :** The value of m is known from some other condition satisfied by the line.

## 5.13 LINE PASSING THROUGH TWO GIVEN POINTS

**To find the equation of the line passing through two given points $(x_1, y_1)$ and $(x_2, y_2)$.**

Let the equation of the line be

$$y = mx + c \qquad \text{....(1)}$$

Since the line passes through the points $(x_1, y_1)$ and $(x_2, y_2)$ these points satisfy equation (1)

$$y_1 = mx_1 + c \qquad \text{....(2)}$$

and $$y_2 = mx_2 + c \qquad \text{....(3)}$$

Subtracting equation (2) from equation (1)

$$y - y_1 = m(x - x_1) \qquad \text{....(4)}$$

Subtracting equation (2) from equation (3)

$$y_2 - y_1 = m(x_2 - x_1)$$

or $$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Putting the value of $m$ in equation (4)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \text{ or } \boxed{\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}}$$

Which is the required equation.

**Note :** The slope of the line joining the points $(x_1, y_1)$ and $(x_2, y_2)$

$$\frac{y_2 - y_1}{y - y_1} = \frac{x_2 - x_1}{x - x_1} \quad m = \boxed{\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}} \Rightarrow \frac{x_2 - x_1}{y_1}$$

$$= \frac{\text{Difference of ordinates of the points}}{\text{Difference of abscissa of the points}}$$



## ILLUSTRATIVE EXAMPLES

**Example 1 : Reduce the equation $3x + 4y = 12$ to (i) Slope intercept form (ii) Intercept form (iii) Normal form. Also find the value of constants used in these standard forms.**

**Solution :** (i) The given equation is

$$3x + 4y = 12 \text{ or } 4y = -3x + 12$$

or $$y = -\frac{3}{4}x + \frac{12}{4} \text{ or } y = -\frac{3}{4}x + 3$$

This is of the form $y = mx + c$, where $m = -\frac{3}{4}$ and $c = 3$

(ii) The given equation is

$$3x + 4y = 12$$

dividing the equation by 12

$$\frac{3x}{12} + \frac{4y}{12} = 1 \text{ or } \frac{x}{4} + \frac{y}{3} = 1$$

Which is of the form $\frac{x}{a} + \frac{y}{b} = 1$, where $a = 4$, $b = 3$.

(iii) The given equation is

$$3x + 4y = 12$$

RHS is positive. Hence, dividing the equation by

$$\sqrt{A^2 + B^2} = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = 5$$

or $$\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$$

Which is the form of $x \cos\alpha + y \sin\alpha = p$, where

$$\cos\alpha = \frac{3}{5}, \sin\alpha = \frac{4}{5}, p = \frac{12}{5}$$

Thus $$\tan\alpha = \frac{4}{3} \text{ and } p = \frac{12}{5}$$

Thus the length of the perpendicular on the line from the origin is $\frac{12}{5}$ units and its inclination with x-axis is $\tan^{-1}\frac{4}{3}$.

**Example 2 : Reduce the equation $\sqrt{3}x - y + 2 = 0$ to normal form. Also find the length of perpendicular on this line from the origin and the inclination of the perpendicular with x-axis.**

**Solution :** The given equation is $\sqrt{3}x - y + 2 = 0$ or $\sqrt{3}x - y = -2$.

Making R.H.S. Positive

$$-\sqrt{3}x + y = 2 \qquad \text{....(1)}$$

Dividing the sides by $\sqrt{A^2 + B^2} = \sqrt{(-\sqrt{3})^2 + (1)^2} = \sqrt{3+1} = 2$

$$-\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 1 \qquad \text{....(2)}$$

This is of the form $x \cos \alpha + y \sin \alpha = p$. Hence, on comparing

$$\cos \alpha = -\frac{\sqrt{3}}{2}, \sin \alpha = \frac{1}{2}, p = 1$$

or $$\cos \alpha = -\frac{\sqrt{3}}{2} = -\cos 30^\circ$$

$$= \cos (180^\circ - 30^\circ) \text{ or } \cos (180^\circ + 30^\circ)$$

$$= \cos 150^\circ \text{ or } \cos 210^\circ$$

Hence, $$\alpha = 150^\circ \text{ or } 210^\circ \qquad \text{.....(3)}$$

Similarly $\sin \alpha = \frac{1}{2} = \sin 30^\circ = \sin (180^\circ - 30^\circ)$

i.e. $$= \sin 30^\circ \text{ or } \sin 150^\circ$$

Hence $$\alpha = 30^\circ \text{ or } 150^\circ \qquad \text{....(4)}$$

The common value of $\alpha$ in equation (3) and (4) is 150° Hence, the length of perpendicular = 1 and its inclination with x-axis is 150°.

**Example 3 : Find the equation of the line passing through the point (3, 2) and making an angle of 60° with x-axis.**

**Solution :** The line makes an angle 60° with x-axis. Hence the slope of the required line $m = \tan 60^\circ = \sqrt{3}$. The given point is (3, 2)

Hence $x_1 = 3$ and $y_1 = 2$. We know that the equation of the line with slope m and passing through the point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1)$$

**Putting the value** $y - 2 = \sqrt{3}(x - 3)$



or $\sqrt{3}x - y + 2 - 3\sqrt{3} = 0$

**Example 4 : Find the equation of line passing through the middle points of the lines joining (4, –7), (–2, 3) and (–4, –7), (–2, – 3)**

**Solution :** The coordinates of the middle point of the line joining (4, –7) and (–2, 3) are

$$= \left(\frac{4+(-2)}{2}, \frac{-7+3}{2}\right) = (1, -2) \qquad \text{....(1)}$$

Similarly the coordinates of middle point of the line joining (–4, –7, (–2, –3) are

$$= \left(\frac{4+(-2)}{2}, \frac{-7-3}{2}\right) = (-3, -5) \qquad \text{.....(2)}$$

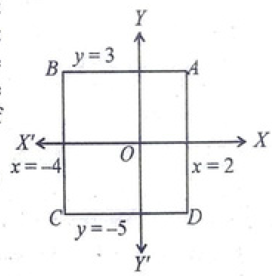
Now, the equation of the line passing through (1, –2) and (–3, –5) is

$$y - (-2) = \frac{(-5)-(-2)}{(-3)-(1)}(x - 1)$$

or $$3x - 4y - 11 = 0$$

**Example 5 : The equations of the sides of a rectangle are $x = 2$, $x = -4$, $y = 3$ and $y = -5$. Find the equations of its diagonals.**

**Solution :** Equations $x = 2$ and $x = -5$ represent lines parallel to $y$-axis and $y = 3$, $y = -5$ represent the lines parallel to $x$-axis. Let these lines rectangle ABCD. Hence, the coordinates of vertices of the rectangle are obtained by solving the equations of the sides. These are




$A(2, 3)$, $B(-4, 3)$, $C(-4, -5)$ and $D(2, -5)$

Hence, the equation of the diagonal $AC$ is

$$y - 3 = \frac{-5-3}{-4-2}(x - 2)$$

or $$4x - 3y = 1 = 0$$

Similarly the equation of the diagonal $BD$ is

$$y - 3 = \frac{-5-3}{2+4}(x + 4)$$

or $$4x + 3y + 7 = 0$$

Hence, the required equations of the diagonals are

$$4x - 3y + 1 = 0, 4x + 3y + 7 = 0$$

**Example 6 : Prove that the points $(a, 0)$, $(0, b)$, $(3a, -2b)$ are collinear. Also find the equation of line.**

**Solution :** The equation of the line passing through $(a, 0)$ and $(0, b)$ is

$$y - 0 = \frac{b-0}{0-a}(x - a)$$

or $$bx + ay = ab \quad \ldots(1)$$

The point $(3a, -2b)$ will also lie on this line if its coordinates satisfy equation (1)

$\therefore \quad b(3a) + a(-2b) = ab$

$3ab - 2ab = ab$

$ab = ab$ which is true.

Thus the point $(3a, -2b)$ satisfies equation (1). Hence, the three points are collinear and the equation of the line is $bx + ay = ab$.

## EXERCISES 5.3

1. Reduce the following equations to slope intercepts and intercept forms. Also find the values of the constants used in these standard forms.
   (i) $3x + 4y = 12$ (ii) $7x - 3y = 15$ (iii) $5x + 6y + 8 = 0$
2. Find the slope of the line $x \cos \alpha + y \sin \alpha = p$.
3. Find the tangent of the angles that the following lines make with the positive direction of x-axis.
   (i) $\sqrt{3}x - y + 2 = 0$
   (ii) $x + \sqrt{3}y - 2\sqrt{3} = 0$
4. Find the length of the intercept of the line $3x + 4y = 6$ between the axes. Also find its middle point.
5. Find the values of $a$ and $b$ such that the equations $5x + 4y = 20$ and $ax - by + 1 = 0$ represent the same line.
6. Reduce the following equations to the form $x \cos \alpha + y \sin a = p$:
   (i) $x + y + \sqrt{2} = 0$
   (ii) $\sqrt{3}x - y + 2 = 0$
7. Find the equation of the line passing through the point (2, 3) and making an angle 45° with $x$-axis.
8. Find the equation of the line passing through the point $(p \cos \alpha, p \sin \alpha)$ and making an angle 90° along $x$-axis.

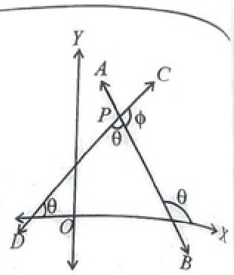


9. Find the equations of lines passing through the following pairs of points.
   (i) (3, 4) and (5, 6) (ii) $(0, -a)$ and $(b, 0)$
   (iii) $(a, b)$ and $(a + b, a - b)$ (iv) $(at_1, a/t_1)$ and $(at_2, a/t_2)$
   (v) $(a \sec \alpha, b \tan \alpha)$ and $(a \sec \beta, b \tan \beta)$
10. The vertices of a triangle ABC are (1, 1), (–2, 0) and (6,4) respectively. Prove that the equation of the median through the vertex A is $x - y = 0$.
11. Find the equations of the sides of the triangle whose vertices are (1,4) (2, 3) and (–1, – 2).
12. Prove that the following sets of points are collinear :
    (i) (1,4) (3, – 2), (– 3, 16) (ii) (3a, 0) (0, 3b), (a, 2b)
    (iii) $(a, b + c)$, $(b, c + a)$, $(c, a + b)$
13. Prove that the points $(a, b)$, $(a', b')$ and $(a - a', b - b')$ are collinear if $ab' = a'b$.

## ANSWERS 5.3

1. (i) $m = -\frac{3}{4}$ $C = 3, a = 4, b = 3$
   (ii) $m = \frac{7}{13}, C = -\frac{15}{13}, a = \frac{15}{7}, b = -\frac{15}{13}$
   (iii) $m = -\frac{5}{6}, C = -\frac{4}{3}, a = -\frac{8}{5}, b = -\frac{4}{3}$
2. $-\cot \alpha$ 3. (i) tan 60° (ii) tan 150°
4. $\frac{5}{2}, \left(1, \frac{3}{4}\right)$ 5. $a = -\frac{1}{4}, b = \frac{1}{5}$
6. (i) $x \cos 225° + y \sin 225 = 1$ (ii) $x \cos 150° + y \sin 150 = 1$
7. $x - y + 1 = 0$ 9. $x \cos \alpha + y \sin \alpha - p = 0$
9. (i) $y - x = 1$ (ii) $ax - by = ab$
   (iii) $(a - 2b)x - by + b^2 + 2ab - a^2 = 0$
   (iv) $t_1 t_2 y + x = a(t_1 + t_2)$
   (v) $bx \cos \frac{\alpha-\beta}{2} - ay \sin \frac{\alpha-\beta}{2} = ab \cos \frac{\alpha+\beta}{2}$
11. $x + 3y + 7 = 10$, $3n - y + 1 = 0$, $7x + y - 11 = 0$

## 5.14 Angle between Two Lines




Let two lines $AB$ and $CD$ make angle $\theta_1$ and $\theta_2$ respectively with the positive direction of x-axis. The two lines intersect at P such that BPD = θ. Now in triangle EPT.

$$\theta + \theta_2 = \theta_1$$

$$\theta = \theta_1 - \theta_2 \quad ......(1)$$

**Case I : When the equations of two lines are in slope intercept form.**

Let the equations of $AB$ and $CD$ in slope intercept form be $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively where $m_1 = \tan\theta_1$, $m_2 = \tan\theta_2$.

From equation (1) $\quad \theta = \theta_1 - \theta_2 \qquad \therefore \tan\theta = \tan(\theta_1 - \theta_2)$

or $\quad \tan\theta = \dfrac{\tan\theta_1 - \tan\theta_2}{1+\tan\theta_1\tan\theta_2}$ or $\tan\theta = \dfrac{m_1 - m_2}{1+m_1m_2}$

Hence, $\quad \theta = \tan^{-1}\left[\dfrac{m_1 - m_2}{1+m_1m_2}\right] \quad ....(2)$

Let the other angle between $AB$ and $CD$ be $\phi$.

Then $\quad \phi = \pi - \theta \qquad \therefore \tan\phi = \tan(\pi - \theta)$

or $\quad$ then $\phi = -\tan\theta$ or $\tan\phi = -\dfrac{m_1 - m_2}{1+m_1m_2}$

Hence $\quad \phi = \tan^{-1}\left[-\dfrac{m_1 - m_2}{1+m_1m_2}\right]$

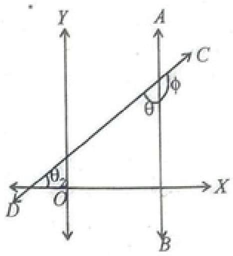
Angle between lines (2) and (3) $= \tan^{-1}\left[\pm\dfrac{m_1 - m_2}{1+m_1m_2}\right]$

**Note :** (i) When the angle between the line is acute then the value of $\dfrac{m_1 - m_2}{1+m_1m_2}$ is taken as positive. When the angle is obtuse, it is taken as negative.

(ii) When one of the two lines is parallel to y-axis then θ cannot be found from the above formulae because the line makes an angle of 90° with x-axis. Hence either $m_1$ or $m_2$ becomes infinite.



Let the line $AB$ be parallel to y-axis i.e. it makes an angle of 90° with x-axis.




From the figure $\quad \theta = \dfrac{\pi}{2} - \theta_2$

or $\quad \tan\theta = \tan\left(\dfrac{\pi}{2} - \theta_2\right)$

or $\quad \tan\theta = \cot\theta_2$

$\Rightarrow \quad \tan\theta = \dfrac{1}{\tan\theta_2} = \dfrac{1}{m_2}$

$\therefore \quad \theta = \tan^{-1}\dfrac{1}{m_2} \quad .....(1)$

Similarly if the other angle between these lines $\angle CPB$ be $\phi$, then

$$\phi = \pi - \theta. \text{ or } \phi = \pi - \left(\frac{\pi}{2} - \theta_2\right) \text{ or } \phi = \frac{\pi}{2} + \theta_2$$

or $\quad \tan\phi = -\dfrac{1}{\tan\theta_2} \Rightarrow \tan\phi = -\dfrac{1}{m_2}$

$\therefore \quad \theta = \tan^{-1}\left(\dfrac{1}{m_2}\right) \quad .....(2)$

Equations (1) and (2) suggest that if one of the two lines is parallel to y-axis, then the angle between the lines is $\tan^{-1}\left(+\dfrac{1}{m_2}\right)$.

**Case II. When the equations of the lines are given in general form :**

Let the equations of the two lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Writing the two equations in slope form.

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1}$$

and $$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2}$$

If $m_1$ and $m_2$ be the slopes of the lines, then

$$m_1 = -\frac{a_1}{b_1} \quad \text{and} \quad m_2 = -\frac{a_2}{b_2}$$

If $\theta$ be the angle between the lines, then $\tan\theta = \pm\frac{m_1 - m_2}{1+m_1m_2}$. Putting the values of $m_1$ and $m_2$

$$\tan\theta = \pm\frac{\left(-\frac{a_1}{b_1}\right)-\left(-\frac{a_2}{b_2}\right)}{1+\left(-\frac{a_1}{b_1}\right)\left(-\frac{a_2}{b_2}\right)}$$

$$\tan\theta = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$$

Hence $$\theta = \tan^{-1}\left(\pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right)$$

**Note :** This formula can be used when one of the lines is parallel to y-axis.

**Case III : When the equation of both the lines are in normal form :**

Let the equations of the two lines in normal form be :

$$x\cos\alpha_1 + y\sin\alpha_1 = p_1 \quad .....(1)$$

$$\text{and } x\cos\alpha_2 + y\sin\alpha_2 = p_2 \quad .....(2)$$

Slope of the line (1) $m_1 = -\frac{coeff.of\ x}{coeff.of\ y} = -\frac{\cos\alpha_1}{\sin\alpha_1}$

or $m_1 = -\cot\alpha_1 \Rightarrow \tan\theta_1 = \tan(90 + \alpha_1)$

(Here $\theta_1$ is the angle made by the line (1) with positive direction of x-axis)

$$\therefore \quad \theta_1 = 90° + \alpha_1 \quad ......(3)$$

Similarly, slope of line (2)

$$m_2 = -\frac{\cos\alpha_2}{\sin\alpha_2} = -\cot\alpha_2$$

$$\tan\theta_2 = \tan(90° + \alpha_2)$$

$$\therefore \quad \theta_2 = 90° + \alpha_2 \quad .....(4)$$



(Here $\theta_2$ is the angle made by the line (2) with positive direction of x-axis)

Hence, the angle between two lines $\theta = \theta_1 \sim \theta_2$

$$= \left(\frac{\pi}{2}+\alpha_1\right) \sim \left(\frac{\pi}{2}+\alpha_2\right)$$

or $$\theta = \alpha_1 \sim \alpha_2 = |\alpha_1 - \alpha_2|$$

## 5.15 Necessary Condition for two Lines to be Parallel

If the two lines are parallel then the angle between them is zero Hence, the value of the tangent of the angle is also zero.

**Case I : When the equations of the two lines are given in slope form :**

Let the equations of the line in slope form be

$$y = m_1x + c_1 \text{ and } y = m_2x + c_2$$

If $\theta$ be the angle between the lines then

$$\tan\theta = \pm\frac{m_1 - m_2}{1+m_1m_2}$$

The lines are parallel. Hence, $\theta = 0$ or $\tan\theta = \tan 0 = 0$. On putting these values

$$\tan 0 = \pm\frac{m_1 - m_2}{1+m_1m_2} = 0 \text{ or } (m_1 - m_2) = 0$$

Hence, if the two lines are parallel, then $m_1 = m_2$ i.e. their slopes are equal.

**Case II : When the equations of the two lines are given in general form :**

Let the equations of the lines in general form be

$$a_1x + b_1y + c_1 = 0 \quad .....(1)$$

$$\text{and} \quad a_2x + b_2y + c_2 = 0 \quad ....(2)$$

If $\theta$ is the angle between these lines, then from the formula

$$\tan\theta = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2} \quad ....(3)$$

If the lines represented by (1) and (2) are parallel then $\theta = 0$, i.e. $\tan\theta = \tan 0 = 0$

$$\therefore \quad \tan 0 = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2} \Rightarrow 0 = a_2b_1 - a_1b_2$$

or $$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

i.e. in equation of parallel lines the coefficients of x and y in two equations are proportional. Hence, the coefficients of x and y in two equations can be made equal by multiplying them by some suitable number.

Hence, we can say that the equations of parallel lines differ by constants. Hence, if the equation of one line is $ax + by + c_1 = 0$ then the equation of parallel line is $ax + by + c_2 = 0$. The value of $c_2$ can be found from the conditions given in the problem.

## 5.16 Condition for two lines to be mutually perpendicular

**Case I : When the equations of the lines are given in slope form.**

Let the equation of line in slope form be

$$y = m_1x + c_1 \qquad \text{......(1)}$$
$$y = m_2x + c_2 \qquad \text{.....(2)}$$

If $\theta$ be the angle between them, then $\tan\theta = \pm\left[\frac{m_1 - m_2}{1 + m_1m_2}\right]$

If these lines are mutually perpendicular,

then $\theta = 90^\circ$ i.e. $\tan\theta = \tan 90^\circ = \infty$

Hence, $1 + m_1m_2 = 0$ or $m_1m_2 = -1$ or $m_2 = -\frac{1}{m_1}$

Hence the product of slopes of two perpendicular lines in $-1$. The slope of the line perpendicular to a given line is obtained by taking the reciprocal of the slope of the given line with changed sign.

**Case II : When the equations of the lines are given in general form :**

Let the equation of the line in general form be

$$a_1x + b_1y + c_1 = 0 \qquad \text{......(1)}$$
$$a_2x + b_2y + c_2 = 0 \qquad \text{.....(2)}$$

we know that if $\theta$ is the angle between the lines, then

$$\tan\theta = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$$

If the lines are perpendicular then $\theta = 90^\circ$



$$\tan 90^\circ = \pm\left[\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}\right] \text{ or } \infty = \pm\frac{a_2b_1 - a_1b_2}{a_1a_2 + b_1b_2}$$

or $$a_1a_2 + b_1b_2 = 0$$

**Note :** The slope of the line $ax + by + c = 0$ is $m_1 = -\frac{a}{b}$ then the slope of a perpendicular line $= \frac{b}{a}$

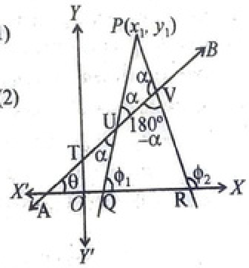
$\therefore$ The equation of the perpendicular line in slope form

$$y = \frac{b}{a}(x) + \lambda$$
$$ay = bx + a\lambda$$
$$bx - ay + a\lambda = 0$$
$$bx - ay + c_1 = 0$$

Where $c_1 = a\lambda$ which is the equation of a line perpendicular to $ax + by + c = 0$.

## 5.17 Equation of a line passing through a given point and making a given angle with the given line

Let $P(x_1, y_1)$ be a given point and AB be a given line making angle $\theta$ with x-axis. Let $PQ$ and $PR$ be two lines which pass through P and make the given angle $\alpha$ with $AB$. Let $PQ$ and $PR$ make angles $\phi_1$ and $\phi_2$ respectively with x-axis.




$\therefore$ Equation of $PQ$ is

$$y - y_1 = \tan\phi_1\,(x - x_1) \quad \text{....(1)}$$

Equation of $PR$ is

$$y - y_1 = \tan\phi_2\,(x - x_1) \quad \text{.....(2)}$$

Now the equation of the given line AB is

$$y = mx + c$$

when $m = \tan\theta$ and $c = OT$

Also AB makes angle $\alpha$ with $PQ$ and $PR$

In $\Delta AUQ$ and $\Delta AVR$,

$$\phi_1 = \theta + \alpha \text{ and } \phi_2 = \theta + (180 - \alpha)$$

Hence, $\tan\phi_1 = \tan(\theta+\alpha) \Rightarrow \tan\phi_1 = \dfrac{\tan\theta+\tan\alpha}{1-\tan\theta\tan\alpha}$

or $$\tan\phi_1 = \frac{m+\tan\alpha}{1-m\tan\alpha} \qquad .....(3)$$

or $\tan\phi_2 = \tan[\theta+(180-\alpha)] = \tan[180^\circ+(\theta-\alpha)]$

Hence $$\tan\phi_2 = \tan(\theta-\alpha) = \frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha} = \frac{m-\tan\alpha}{1+m\tan\alpha} \qquad .....(4)$$

Putting the values of the $\tan\phi_1$ and $\tan\phi_2$ from equations (3) and (4) in equations (1) and (2) the equation of the required lines are.

$$y - y_1 = \frac{m+\tan\alpha}{1-m\tan\alpha}(x-x_1) \qquad ....(5)$$

or $$y - y_1 = \frac{m-\tan\alpha}{1+m\tan\alpha}(x-x_1) \qquad .....(6)$$

## 5.18 Condition of Concurrency of three straight lines

If three lines pass through one point, they are said to be concurrent. Hence, the point of intersection of any two lines lies on the let the equations of three lines be

$$a_1x + b_1y + c_1 = 0 \qquad ......(1)$$
$$a_2x + b_2y + c_2 = 0 \qquad ......(2)$$
$$a_3x + b_3y + c_3 = 0 \qquad ......(3)$$

The following methods are used to find the conditions of concurrency.

**First method :** We solve the equations of any two of the given lines to find the coordinates of point of intersection. We then put these coordinates in the equation of the third line. If the lines are concurrent the coordinates of the point of intersection will satisfy the equation of the third line.

$\because$ The coordinates of point of intersection of (1) and (2) are

$$\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$$

These coordinates will satisfy equation (3) if

$$a_3\cdot\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}\right) + b_3\left(\frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right) + c_3 = 0$$



i.e. it $a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1) = 0$

Hence, $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$

is the required condition :

This condition can be represented in determinant from as :

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

**Second method :** The three lines will be concurrent it they pass through one point say $(x_1, y_1)$

$\because$ The point $(x_1, y_1)$ lies on all the three lines, hence it will satisfy the equation of three lines.

Hence $$a_1x_1 + b_1y_1 + c_1 = 0 \qquad .....(4)$$
$$a_2x_1 + b_2y_1 + c_2 = 0 \qquad .....(5)$$
$$a_3x_1 + b_3y_1 + c_3 = 0 \qquad .....(6)$$

Eliminating $x_1, y_1$ from equations (4), (5) and (6)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

which is the required condition.

**Third method :** If we can find three constants $p, q, r$ such that for all values of $x$ and $y$

$$p(a_1x + b_1y + c_1) + q(a_2x + b_2y + c_2) + r(a_3x + b_3y + c_3) = 0 \qquad .....(7)$$

Then the three lines are concurrent because in such case

$$a_2x + b_3y + c_3 = -\frac{p}{r}(a_1x + b_1y + c_1) - \frac{q}{r}(a_2x + b_2y + c_2) \qquad ...(8)$$

If the point of intersection of first two line is $(x_1, y_1)$ then

$$a_1x_1 + b_1y_1 + c_1 = 0 \text{ and } a_2x_1 + b_2y_1 + c_2 = 0$$

Putting $(x_1, y_1)$ in equation (8)

$$a_3x_1 + b_3y_1 + c_3 = -\frac{p}{r}(a_1x_1 + b_1yl + c_1) - \frac{p}{r}(a_2x_1 + b_2y_1 + c_2)$$

$$= -\frac{p}{r}(0) - \frac{p}{r}(0) = 0$$

Hence $a_3x_1 + b_3y_1 + c_3 = 0$

From this we conclude that the point of intersection of lines (1) and (2) lies on the line (3). Hence the lines are concurrent.

## ILLUSTRATIVE EXAMPLES

**■ Example 1. Find the angle between the lines $3x + y - 7 = 0$ and $x + 2y - 9 = 0$**

**Solution :** The slope of the given line $3x + y - 7 = 0$

$$m_1 = -\frac{coeff.of\ x}{coeff.of\ y} = -\frac{3}{1}$$

Similarly, the slope of the other line $x + 2y + 9 = 0$ is

$$m_2 = -\frac{coeff.of\ x}{coeff.of\ y} = -\frac{1}{2}$$

Let $\theta$ be the angle between the lines.

Hence, $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{-3 + ½}{1 + (-3)(-½)} = \pm(-1)$

Hence, $\tan\theta = -1$ and $1$

$\therefore$ $\tan\theta = -\tan 45°$ and

and, $\tan\theta = \tan 45$

$\therefore$ $\theta = 135°$ and $\theta = 45°$

**■ Example 2. Find the equation of the line passing through the point (1, 1) and parallel to the line $3x - 4y = 7$.**

**Solution :** First method : Equation of line through (1, 1) is

$$y - 1 = m(x - 1) \quad .....(1)$$

The slope of the given line $3x - 4y = 7$ .....(2)

is $m_1 = -\frac{coeff.of\ x}{coeff.of\ y} = \frac{3}{4}$ ....(3)

Lines (1) and (2) are parallel

$$m = m_1 = \frac{3}{4}$$

Putting value of m in (2)



$$y - 1 = \frac{3}{4}(x - 1)$$

or $3x - 4y + 1 = 0$

Which is the equation of the required line

**Second Method :** Equations of a line parallel to the line

$3x - 4y = 7$ is $3x - 4y = \lambda$ ....(1)

Where $\lambda$ is an arbitrary constant

This passes through (1, 1)

Thus, $3 \times 1 - 4 \times 1 = \lambda \Rightarrow \lambda = -1$

Putting the value of $\lambda$ in (1), $3x - 4y + 1 = 0$

Which is the equation of the required line.

**■ Example 3. Prove that the lines $2x - y + 9 = 0$ and $4x - 2y - 8 = 0$ are parallel.**

**Solution :** Equation of first line is $2x - y + 9 = 0$ whose slope is

$\therefore$ $m_1 = -\frac{coeff.of\ x}{coeff.of\ y} = 2$ ....(1)

Similarly, equation of other line is $4x - 2y - 8 = 0$ whose slope is

$\therefore$ $m_2 = -\frac{coeff.of\ x}{coeff.of\ y} = 2$ .....(2)

Hence $m_1 = m_2$

Thus the slopes of the two lines are equal. Hence, the two lines are parallel.

**■ Example 4 : Find the equation of the line which passes through the point (1, 2) and is parallel to the line joining the points (4, –3) and (2, 5).**

**Solution :** The line with slope $m$ and passing through (1, 2) is

$$y - 2 = m(x - 1) \quad ....(1)$$

Slope of the line joining (4, –3) and (2, 5) is

$$m_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{5 - (-3)}{2 - 4} = -4$$

The line (1) will be parallel to the line joining (4, –3) and (2, 5) if

$$m = m_1$$

$$m = -4$$

Putting the value of $m$ in equation (1)

$$y-2=-4(x-1)$$

or $$4x+y-6=0$$

Which is the equation of the required line.

**Example 5 : Find the equation of the line passing through the point (3, –4) and parallel to x-axis.**

**Solution :**

**First method :** Equation of a line through $(3,-4)$

is $$y+4=m(x-3) \quad ....(1)$$

We know the slope of x-axis $m=\tan 0=0$. Putting the values of $m$ in eq (1), we get

$$y+4=0(x-3)$$

$$y+4=0$$

$$y=-4$$

**Second method :**

We know that the equation of a line parallel to x-axis and at a distance c is

$$y=c \quad ....(1)$$

This line passes through (3, –4). Hence – 4 = c. Putting the value of c in eq (1) we get the equation of the required line as $y=-4$

**Example 6 : Prove that the lines $2x-y+9=0$ and $x+2y-7=0$ are perpendicular to each other.**

**Solution :** The slope of the line $2x-y+9=0$ is

$$m_1=-\frac{coeff\ of\ x}{coeff\ of\ y}=-\frac{2}{(-1)}=2 \quad ....(1)$$

Similarly, slope o the other line $x+2y-7=0$ is

$$m_2=-\frac{coeff\ of\ x}{coeff\ of\ y}=-\frac{1}{2} \quad ....(2)$$

We know that if two lines are perpendicular if product of their slopes is – 1

Here $\quad m_1\times m_2=2\times(-½)=-1$

Hence the given lines are mutually perpendicular.

**Example 7 : Find the equation of line passing through (3, 2) and perpendicular to the line $y=x$.**

**Solution :** The equation of a line through the point (3, 2) is $(y-2)=m(x-3)$, where m is the slope. The slope of the given line $x-y=0$ is $m_1=\frac{-1}{-1}=1$

Slope of the line perpendicular to the line

$$m=-\frac{1}{m_1}=-1$$

Putting the value of m, we get the equation of the required line

$$y-2=-1(x-3) \text{ or } x+y-5=0$$



**Example 8 : Find the equation of the perpendicular bisector of the line joining the points (2, 1) (4, 3)**

**Solution :** The slope of line joining the points (2, 1) and (4, 3)

$$m_1=\frac{y_2-y_1}{x_2-x_1}=\frac{3-1}{4-2}=1$$

Slope of is perpendicular line

$$m=-\frac{1}{m_1}=-1$$

If $(x_1, y_1)$ be the middle point of line joining (2, 1) and (4, 3) then

$$x_1=\frac{2+4}{2}, y_1=\frac{1+3}{2}$$

$$x_1=3, y_1=2$$

Hence, equation of a line with slope –1 and passing through (3, 2) is

$$y-2=-1(x-3)$$

$$x+y-5=0$$

Which is the required equation.

**Example 9 : Find the equation of the line joining the point (2, –9) and the point of intersection of lines $2x+5y-8=0$ and $3x-4y-35=0$**

**Solution :** Solving the equations $2x+5y-8=0$, $3x-4y-35=0$ the point of intersection is obtained as $x=9, y=-2$

Now the equation of the line joining the given point (2, –9) and the point of intersection (9, –2) is

$$y-(-9)=\frac{(-2)-(-9)}{9-2}(x-2)$$

or $$x-y-11=0$$

Which is the required equation.

**Example 10 : Find the coordinates of the foot of the perpendicular from the point (2, 3) on the line $y = 3x + 4$.**

**Solution :** The equation of the line is $y = 3x + 4$

$$y - 3x - 4 = 0 \quad \text{....(1)}$$

Equation of a line perpendicular to this line is

$$x + 3y + \lambda = 0 \quad \text{....(2)}$$

According to the question, the line passes through (2, 3), Hence

$$2 + 3 \times (3) + \lambda = \Rightarrow \lambda = -11$$

Putting the value of $\lambda$ in equation (2)

$$x + 3y - 11 = 0$$

on solving equation (1) and (3) $x = -\frac{1}{10}$ and $y = \frac{37}{10}$

The coordinates of the foot of the perpendicular are $\left(-\frac{1}{10}, \frac{37}{10}\right)$.

**Example 11 : Find the equations of lines passing through the point of intersection of lines $3x - 4y + 6 = 0$ and $4x - y - 5 = 0$ and cutting off equal intercepts from the axes.** **[R.U. 2015]**

**Solution :** The equations of given lines are

$$3x - 4y + 6 = 0 \text{ and } 4x - y - 5 = 0$$

Their point of intersection, is $x = 2$ and $y = 3$. We know that the equation of the line in intercept from is

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \text{.....(1)}$$

According to the problem the intercepts on the axes are equal (say $a$). Hence, putting b = a in the above equation.

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ or } x + y = a \quad \text{....(2)}$$

This line passes through (2)

$$\therefore \quad a = 5$$

Hence from equation (4)

$$x + y = 5$$

Similarly putting $b = -a$ in eq. (1) we get

$$x - y = a$$



$\because$ This also passes through (2, 3) hence $a = -1$

$\therefore$ Putting in equation (6)

$$x - y + 1 = 0$$

Hence, the equation of the required lines are

$$x + y = 5 \text{ and } x - y + 1 = 0$$

**Example 12 : Find the angle between straight lines $3x + y + 12 = 0$ and $x = 2y - 1 = 0$** **[R.U. 2015]**

**Solution :** The slope of the given line $3x + y - 7 = 0$

$$m_1 = -\frac{\text{coeff.of } x}{\text{coeff.of } y} = -\frac{3}{1}$$

Similarly, the slope of the other line $x + 2y + 9 = 0$ is

$$m_2 = -\frac{\text{coeff.of } x}{\text{coeff.of } y} = -\frac{1}{2}$$

Let $\theta$ be the angle between the lines.

Hence, $\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{-3 + ½}{1 + (-3)(-½)} = \pm(-1)$

Hence, $\tan\theta = -1$ and 1

$\therefore \quad \theta = 135°$ and $\theta = 45°$

**Example 13 : Find locus of the point which has distance 6 from the point (3, 7).** **[R.U. 2016]**

**Solution :** Let given point be $P(\alpha, \beta)$ and other point is Q (3, 7)

$\because \quad PQ = 6 \Rightarrow PQ^2 = 36$

$\Rightarrow \quad (\alpha - 3)^2 + (\beta - 7)^2 = 36$

$\Rightarrow \quad \alpha^2 + \beta^2 - 6\alpha - 14\beta + 22 = 0$

$\therefore$ Locus of $(\alpha, \beta)$ is $x^2 + y^2 - 6x - 14y + 22 = 0$

## EXERCISES 5.4

1. Find the angles between the following lines :
   (i) $y = (2 - \sqrt{3})x + 5$ and $y = (2 + \sqrt{3})x - 7$
   (ii) $2y - 3x + 5 = 0$ and $4x + 5y + 8 = 0$
   (iii) $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} - \frac{y}{a} = 1$
2. Prove that the following lines are parallel.
   (i) $2y = mx + C$ and $4y = 2mx$
   (ii) $x\cos\alpha + y\sin\alpha = p$ and $x + y\tan\alpha = 5\tan\alpha$

3. Prove that the lines with equation $4x + 5y + 7 = 0$ and $5x - 4y - 11 = 0$ are mutually perpendicular.
4. Find the equations of lines which
   (i) Passes through (4, 5) and is parallel to the line $2x - 3y - 5 = 0$
   (ii) Passes through (1, 2) and is perpendicular to the line $4x + 3y + 8 = 0$
   (iii) Is parallel to the line $2x + 5y = 7$ and passes through the middle point of the line joining the points (2, 7) and (–4, 1).
   (iv) divides the line joining the points (–3, 7) and (5 – 4) in the ratio 4 : 7 and is perpendicular to this line.
5. The vertices of a triangle are (0, 0), (4, –6) and (1, –3). Find the equations of perpendiculars drawn from these vertices on the opposite sides of the triangle.
6. Find the equation of the perpendicular bisector of the line segment joining the points (2, –3) and (–1, 5).
7. Find the equation of the line which is perpendicular to the line $\frac{x}{a} - \frac{y}{b} = 1$ from the point where the line meets the x-axis.
8. Find the equation of the line which is parallel to the line $2x + 3y + 11 = 0$ such that the sum of intercepts made by the line on the axes is 15.
9. Find the equation of the line passing through the point (2, –3) and making an angle of 45° with the line $3x - 2y = 4$.
10. Find the equations of the lines which pass through the point (4, 5) and make equal angles with the lines $3x = 4y + 7$ and $5y = 12x + 6$.
11. Find the equations of two lines which pass through the point (3, –2) and make an angle of 60° with the line $x + \sqrt{3}\,y = 1$.
12. Find the points of intersection of the following pairs of lines :
    (i) $3x - 4y - 7 = 0$ and $2x - 7y + 4 = 0$
    (ii) $2x + 3y + 4 = 0$ and $4x + 3y + 2 = 0$
13. Prove that the lines represented by the following equation are concurrent
    (i) $3x - 4y = 13$; $8x - 11y = 33$ and $2x - 3y = 7$
    (ii) $(b + c)x + ay = d$; $(c + a)x + by = d$ and $(a + b)x + cy = d$
    (iii) $\frac{x}{a} + \frac{y}{b} = 1$; $\frac{x}{b} = \frac{y}{a} = 1$, and $y = x$
14. Find the coordinates of the foot of the perpendicular drawn from the point (–1, 2) on the line $(4x - 3y + 5) = 0$.
15. Find the equation of the line passing through the point of intersection of the lines $y = x + 7$ and $x + 2y + 5 = 0$ parallel to the line $5x - 2y + 1 = 0$.



16. Find the equation of the line passing through the point of intersection of the lines $3x - y = 2$ and $y + 2x = 3$ and making an angle of 45° with the x-axis.

## ANSWERS 5.4

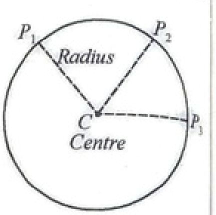
1. (i) 120° or 60° (ii) $\tan^{-1}\left(-\frac{23}{2}\right)$ (iii) 90°
4. (i) $2x - 3y + 7 = 0$ (ii) $3x - 4y + 5 = 0$
   (iii) $2x + 5y = 18$ (iv) $88x - 121y + 371 = 0$
5. $y - x = 0$, $x - 3y = 22$, $2x - 3y = 11$
6. $6x - 16y + 13 = 0$ 7. $ax + by = a^2$
8. $2x + 3y - 18 = 0$ 9. $y + 5x - 7 = 0$ and $5y - x + 17 = 0$
10. $9x - 7y = 1$ and $7x + 9y = 73$
11. $x - \sqrt{3}\,y - 3 - 2\sqrt{3} = 0$ and $x - 3 = 0$
12. (i) (5, 2) (ii) (1, –2)
14. $\left(-\frac{1}{5}, \frac{7}{5}\right)$ 15. $5x - 2y + 33 = 0$
16. $x = y$

Chapter

# 6 Circle, Parabola and Ellipse

## 6.1 Circle

A circle is the set of all points in a plane, each of which is at a constant distance from a fixed point in the plane. In other words, a circle is the locus of a point which moves in a plane so that it remains at a constant distance from a fixed point in the plane. The fixed point is called the centre and the constant distance is called radius is always positive.



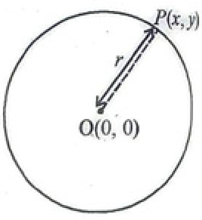

If $P_1$, $P_2$, $P_3$....are points on the circle with centre C, and radius r, then

$$CP_1 = CP_2 = CP_3 = ...... = r.$$

Equation of a circle is simplest if its centre is at the origin.

### Standard (or simplest) form

Let O(0, 0) be the centre of the circle and r(> 0) be its radius. Let $P(x, y)$ be a point in the plane, then P lies on the circle iff




$$OP = r$$

$$\Rightarrow \quad \sqrt{(x-0)^2 + (y-0)^2} = r$$

$$\Rightarrow \quad \boxed{x^2 + y^2 = r^2}$$

which is the equation of the circle.

This is known as standard (or simplest) form.

### Central form

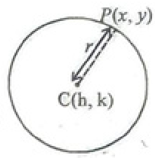
Let $C(h, k)$ be the centre of the circle and $r(>0)$ be its radius. Let $P(x, y)$ be a point in the plane, then $P$ lies on the circle iff



$$CP = r$$

$$\Rightarrow \quad \sqrt{(x-h)^2 + (y-k)^2} = r$$

$$\Rightarrow \quad (x-h)^2 + (y-k)^2 = r^2$$



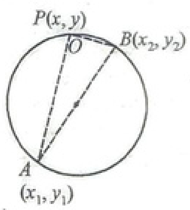

which is the equation of the circle. This is known as Central form.

### Diameter form

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be the extremities of a diameter of the circle, Let $P(x, y)$ different from A and B, be a point on the circle then

Slope of line AP $= \dfrac{y-y_1}{x-x_1}$ and

Slope of line BP $= \dfrac{y-y_2}{x-x_2}$




Now P will lie on the circle iff $\angle APB = 90°$

$$\Rightarrow \quad \frac{y-y_1}{x-x_1} \cdot \frac{y-y_2}{x-x_2} = -1$$

$$\Rightarrow \quad (y-y_1)(y-y_2) = -(x-x_1)(x-x_2)$$

$\Rightarrow \boxed{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) = 0,}$ which is the equation of the circle. This is known as diameter form.

### General Form

We know that the equation of the circle with centre $(h, k)$ and radius $r(>0)$ is

$$(x-h)^2 + (y-k)^2 = r^2 \quad ....(i)$$

$$\Rightarrow \quad x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0 \quad .....(ii)$$

it can be witten as

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where $g = -h, f = -k$ and $c = h^2 + k^2 - r^2$ such that

$$g^2 + f^2 - c = (-h)^2 + (-k)^2 - (h^2 + k^2 - r^2) = r^2 > 0 \quad (\because r > 0)$$

conversely if we consider any equation $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(iii)

with $g^2 + f^2 - c$ (>0, then on adding $g^2 + f^2$ to both sides of (iii), we get

$$(x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) + c = g^2 + f^2$$

$$\Rightarrow (x+g)^2 + (y+f)^2 = g^2 + f^2 - c$$

$\Rightarrow (x-(-g))^2 + (y-(-f))^2 = \left(\sqrt{g^2+f^2-c}\right)^2$

$\Rightarrow \sqrt{(x-(-g))^2+(y-(-f))^2} = \sqrt{g^2+f^2-c}$

$\Rightarrow$ The distance of the point (x, y) from the point $(-g, -f)$ is a fixed positive real number $r = \left(\sqrt{g^2+f^2-c}\right)$ $(\because g^2+f^2-c>0 \Rightarrow \sqrt{g^2+f^2-c}$ is a real number)

$\Rightarrow$ The locus of (iii) is the set of the all points $(x, y)$ which are at a constant distance r $\left(=\sqrt{g^2+f^2-c}\right)$ from the fixed point $(-g-f)$

$\Rightarrow$ The equation (iii) represents a circle with centre $(-g-f)$ and radius $\sqrt{g^2+f^2-c}$.

Thus, we have proved that the equation $x^2+y^2+2gx+2fy+c=0$ represents a circle iff $g^2+f^2-c>0$ its centre is $(-g, -f)$ and radius $= \sqrt{g^2+f^2-c}$. This is known as general form.

## Illustrative Examples

solve

**Example 1 : Find the equation of a circle whose centre is (–2, 3) and radius is 4.**

**Solution :** Since the centre of the circle is (–2, 3) and its radius is 4, therefore, the equation of the circle is

$$(x-(-2))^2 + (y-3)^2 = 4^2$$
$$\Rightarrow \quad (x+2)^2 + (y-3)^2 = 16$$
$$\Rightarrow \quad x^2+4x+4+y^2-6y+9 = 16$$
$$\Rightarrow \quad x^2+y^2+4x-6y-3=0$$

solve

**Example 2 : Find the equation of the circle with centre (2, 2) and which passes through the point (4, 5).**

**Solution :** The centre of the circle is $C(2, 2)$ and it passes through the point $P(4, 5)$

$\therefore$ Radius of circle $= CP$

$$= \sqrt{(4-2)^2+(5-2)^2}$$



$$= \sqrt{4+9} = \sqrt{13}.$$

$\therefore$ The equation of the circle is

$$(x-2)^2 + (y-2)^2 = \left(\sqrt{13}\right)^2 \qquad \text{(Central form)}$$

i.e. mest $x^2+y^2-4x-4y=5$.

**Example 3 : Find the equation of a circle whose Centre is (3, –2) and which passes through the intersection of the lines $5x+7y=3$ and $2x-3y=7$.**

**Solution :** Given lines are

$5x+7y-3=0$ .....(i) and $2x-3y-7=0$ ....(ii)

Solving (i) and (ii) simultaneously, we get $x=2, y=-1$

$\therefore$ The point of intersection, say P of the given lines is (2, –1).

Since the centre of the circle is $C(3, -2)$ and it passes through the point $P(2, -1)$.

it radius $= CP = \sqrt{(2-3)^2+(-1+2)^2} = \sqrt{1+1} = \sqrt{2}$

$\therefore$ The equation of the circle is

$$(x-3)^2 + (y+2)^2 = \left(\sqrt{2}\right)^2 \qquad \text{(central form)}$$

i.e. $x^2+y^2-6x+4y+11=0$

**Example 4 : Find the equation of the circle which passes through the point (–2, –3) and has its centre on the negative direction of x-axis and is of radius 5 units.**

**Solution :** As the centre of the circle lies on the negative direction of x-axis, let its centre be $C(h, 0)$ $h<0$.

Since the circle passes through $A(-2, -3)$ and has radius 5, $CA=5$

$$\Rightarrow \quad (h+2)^2 + (0+3)^2 = 5^2$$
$$\Rightarrow \quad (h+2)^2 = 25-9 = 16 \Rightarrow h+2 = 4, -4$$
$$\Rightarrow \quad h = 2, -6 \quad \text{but} \quad h<0 \Rightarrow h=-6.$$

$\therefore$ The centre of the circle is (–6, 0) and hence its equation is

$$(x+6)^2 + (y-0)^2 = 5^2 \qquad \text{(Central form)}$$

i.e. $x^2+y^2+12x+11=0$

**Example 5 : Find the equation of the circle with radius 5 whose centre lies on x -axis and passes through the point (2, 3).**

**Solution :** As the centre of the circle lies on x-axis, let its centre be C (h , 0).

Since the circle passes through $A(2, 3)$ and has radius

$$CA = 5$$

$$\Rightarrow \quad (2-h)^2 + (3-0)^2 = 5^2$$

$$\Rightarrow \quad (2-h)^2 = 16$$

$$\Rightarrow \quad 2-h = 4, -4$$

$$\Rightarrow \quad h = -2, 6$$

$\therefore$ The centre of the circle is $(-2, 0)$ or $(6, 0)$.

The equation of the circle is

$$(x+2)^2 + (y-0)^2 = 5^2 \text{ or } (x-6)^2 + (y-0)^2 = 5^2$$

i.e. $\quad x^2 + y^2 + 4x - 21 = 0$ or $x^2 + y^2 - 12x + 11 = 0$

There are two circles satisfying the given conditions.

**■ Example 6 : Find the equation of the circle if the end points of a diameter are $A(-2, 3)$ and $B(3, -5)$.**

**Solution :** Using diameter form, the equation of the circle having $A(-2, 3)$ and $B(3, -5)$ as the and points of a diameter is

$$(x-(-2))(x-3) + (y-3)(y-(-5)) = 0$$

or $\quad (x+2)(x-3) + (y-3)(y+5) = 0$

or $\quad x^2 - x - 6 + y^2 + 2y - 15 = 0$

or $\quad x^2 + y^2 - x + 2y - 21 = 0$

**■ Example 7 : Find the equation of the circle drawn on a diagonal of the rectangle as its diameter whose sides are the lines $x = 4$, $x = -5$, $y = 5$ and $y = -1$.**

**Solution :**

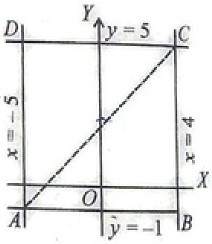
Let $ABCD$ be the rectangle formed the lines.

$x = 4$ ....(i)

$x = -5$ ....(ii)

$y = 5$ .....(iii)

$y = -1$ .....(iv)




The coordinates of the points A, B, C. and D are $(-5, -1)$, $(4-1)$, $(4, 5)$ and $(-5, 5)$ respectively.

The equation of the circle having the diagonal $AC$ as its diameter is

$$(x-(-5))(x-4) + (y-(-1))(y-5) = 0 \quad \text{(Using diameter form)}$$

or $\quad (x+5)(x-4) + (y+1)(y-5) = 0$

or $\quad x^2 + y^2 + x - 4y - 25 = 0$



**■ Example 8 : Find the centre and the radius of the circle $x^2 + y^2 - 8x + 10y - 12 = 0$**

**Solution :** The given equation can be written as

$$(x^2 - 8x) + (y^2 + 10y) = 12$$

Completing the squares, we get

$$(x^2 - 8x + 16) + (y^2 + 10y + 25) = 12 + 16 + 25$$

$$\Rightarrow \quad (x-4)^2 + (y+5)^2 = 53$$

$$\Rightarrow \quad (x-4)^2 + (y-(-5))^2 = \left(\sqrt{53}\right)^2$$, which is comparable with

$$(x-h)^2 + (y-k)^2 = r^2$$

Hence $\quad h = 4, k = -5$, and $r = \sqrt{53}$

**■ Example 9 : Find the centre and the radius of the circle $2x^2 + 2y^2 = x$**

**Solution :** The given equation is $2x^2 + 2y^2 = x$

it can be written as $\quad x^2 + y^2 = \frac{x}{2}$

$$\Rightarrow \quad \left(x^2 - \frac{x}{2}\right) + y^2 = 0$$

Completing the square, we get

$$\left(x^2 - \frac{x}{2} + \frac{1}{16}\right) + y^2 = \frac{1}{16}$$

$$\Rightarrow \quad \left(x - \frac{1}{4}\right)^2 + (y-0)^2 = \left(\frac{1}{4}\right)^2$$, which is comparable with

$$(x-h)^2 + (y-k)^2 = r^2$$

Here $\quad h = \frac{1}{4}$, k = 0 and $r = \frac{1}{4}$

Hence, the given circle has centre at $\left(\frac{1}{4}, 0\right)$ and its radius $= \frac{1}{4}$

**■ Example 10 : Which of the following equations represent a circle ? If any determine its centre and radius :**

(i) $3x^2 + 3y^2 + 6x = 4y + 1$

(ii) $x^2 + y^2 - 12x + 6y + 45 = 0$

**Solution :** (i) The given equation is $3x^2 + 3y^2 + 6x - 4y - 1 = 0$. It can be written as ....(i)

$$x^2 + y^2 + 2x - \frac{4}{3}y - \frac{1}{3} = 0 \quad ....(ii)$$

which is comparable with $x^2 + y^2 + 2gx + 2fy + c = 0$

Here $g = 1, f = -\frac{2}{3}$ and $c = -\frac{1}{3}$

$\therefore$ $g^2 + f^2 - c = 1 + \frac{4}{9} + \frac{1}{3} = \frac{16}{9} > 0$

Hence, the given equation represents a circle with centre $\left(-1, \frac{2}{3}\right)$

and radius $= \sqrt{g^2 + f^2 - c} = \sqrt{\frac{16}{9}} = \frac{4}{3}$

(ii) The given equation is $x^2 + y^2 - 12x + 6y + 45 = 0$ ....(i)

which is comparable with $x^2 + y^2 + 2gx + 2fy + c = 0$

Here $g = -6, f = 3$ and $c = 45$

$\therefore$ $g^2 + f^2 - c = 36 + 9 - 45 = 0$

Hence, the given equation represents a point circle with centre $(6, -3)$ and radius zero.

## EXERCISES 6.1

1. Find the equation of the circle whose.
   (i) Centre is at the origin and the radius is 5 units.
   (ii) Centre is (0, 2) and radius 2 (iii) Centre is (–3, 2) and radius 4
   (iv) Centre is (1, 1) and radius $\sqrt{2}$ (v) Centre is $\left(\frac{1}{2}, \frac{1}{4}\right)$ and radius $\frac{1}{12}$
   (vi) Centre is (–9, –6) and radius $\sqrt{a^2 - b^2}$ (0, 0)
2. Write the equation of a circle whose centre is at origin and which passes through the point (3, –4).
3. Determine the equation of a circle whose centre is (8, –6) and which passes through the point (5, –2).
4. Find the equation of a circle of radius 6 units and whose centre lies in the negative direction of x-axis at a distance of 4 units from the origin.

Note: If let any point that means (x, y) centre means (h, k)



5. Find the equation of the circle whose centre lies in the negative direction of y-axis at a distance 3 units from origin and whose radius is 4 units.
6. Find the equation of the circle which has $A(1, 3)$ and $B(4, 5)$ as opposite ends of a diameter.
7. Find the equation of the circle which has the points (–2, 3) and (0, –1) as opposite ends of a diameter.
8. Find the equation of the circle which passes through the origin and cuts off intercepts –2 and 3 from the coordinate axis.
9. Find the equation of the circle which passes through origin and cuts of intercepts 3 and –2 from the coordinate axis.
10. Find the centre and the radius of the circle :
    (i) $(x + 5)^2 + (y - 3)^2 = 36$ (ii) $\left(x - \frac{1}{2}\right)^2 + \left(y + \frac{1}{3}\right)^2 = \frac{1}{4}$
    (iii) $x^2 + y^2 + 6x + 10y - 8 = 0$ (iv) $x^2 + y^2 - 4x - 8y - 45 = 0$
    (v) $2x^2 + 2y^2 - 3x + 5y - 7 = 0$ (vi) $x^2 + y^2 - 4x - by = 0$
11. Which of the following equations represent a circle if so, determine its centre and radius.
    (i) $x^2 + y^2 + x - y = 0$ (ii) $2x^2 + 2y^2 = 5x + 7y + 3$
    (iii) $x^2 + y^2 + 2x + 10y + 26 = 0$
12. Find the equation of the circle whose centre is (2, –3) and passes through the intersection of the lines $3x - 2y - 1 = 0$ and $4x + y - 27 = 0$.
13. Find the equation of the circle which passes through the point (2, 4) and centre at the intersection of the lines $x - y = 4$ and $2x + 3y + 17 = 0$.

## ANSWERS 6.1

1. (i) $x^2 + y^2 = 25$ (ii) $x^2 + y^2 - 4y = 0$
   (iii) $x^2 + y^2 + 6x - 4y - 3 = 0$ (iv) $x^2 + y^2 - 2x - 2y = 0$
   (v) $36x^2 + 36y^2 - 36x - 18y + 11 = 0$
   (vi) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
2. $x^2 + y^2 = 25$
3. $x^2 + y^2 - 16x + 2y + 75 = 0$
4. $5x^2 + y^2 + 8x - 20 = 0$
5. $x^2 + y^2 + 6y - 7 = 0$
6. $x^2 + y^2 - 5x - 8y + 19 = 0$
7. $x^2 + y^2 + 2x - 2y - 3 = 0$
8. $x^2 + y^2 + 2x - 3y = 0$
9. $x^2 + y^2 - 3x + 2y = 0$

10. (i) $(-5, 3); 6$ (ii) $\left(\frac{1}{2}, -\frac{1}{3}\right); \frac{1}{2}$

(iii) $(-9, -5); 7$ (iv) $(2, 9); \sqrt{65}$

(v) $\left(\frac{3}{4}, -\frac{5}{4}\right); \frac{3}{4}\sqrt{10}$ (vi) $\left(\frac{a}{2}, \frac{b}{2}\right); \frac{1}{2}\sqrt{a^2+b^2}$

11. (i) circle; $\left(-\frac{1}{2}, \frac{1}{2}\right); \frac{1}{\sqrt{2}}$ (ii) circle; $\left(\frac{5}{9}, \frac{7}{4}\right), \frac{7}{4}\sqrt{2}$

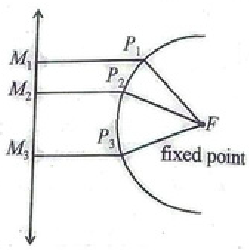
(iii) Point circle, $(-1, -5)$zero

12. $x^2 + y^2 - 4x + 6y - 96 = 0$

13. $x^2 + y^2 - 2x + 5y - 40 = 0$

## 6.2 Parabola

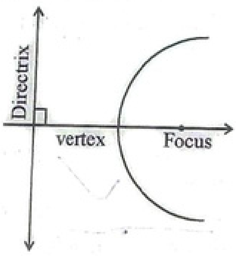
A parabola is the set of all points in a plane which are equidistant from a fixed line and a fixed point (not on the line) in the plane.




The fixed line (say $l$) is called the directrix of the parabola and the fixed point (say F) is called the focus of the parabola.

If $P_1$, $P_2$, $P_3$ are points on the parabola and $M_1P_1$, $M_2P_2$, $M_3P_3$ are are Perpendicular to the directrix then $FP_1 = M_1P_1$, $FP_2 = M_2P_2$, $FP_3 = M_3P_3$ etc.

The line passing through the focus and perpendicular to the direction is called the axis of parabola. The point of intersection of parabola with its axis is called vertex of parabola.




The equation of a parabola is in simplest from if its vertex is at the origin and its axis along either x-axis or y-axis.

### To find the equation of a parabola in the standard form $y^2 = 4ax$, $a > 0$

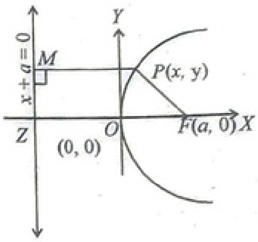
Let F be the focus, $l$ be the directrix and $Z$ be the foot of perpendicular from F



to the line $l$. Take ZF as x-axis with positive direction from Z to F. Let O be the midpoint of ZF, take O as origin, then the line through O and perpendicular to ZF becomes y-axis.

Let $ZF = 2a$

Then $ZO = OF = a$




Since F lies to the right of $O$ and $Z$ lies to the left of $O$, coordinate of $F$, $Z$ are $(a, 0)$, $(-a, 0)$ respectively. Therefore the equation of the line t i.e. direction is $x = -a$ i.e. $x + a = 0$.

Let P(x, y) be any point in the plane of the line L and point F, and MP be the perpendicular distance from P to the line $l$ then P lies on parabola iff.

$$FP = MP$$

$$\Rightarrow \quad \sqrt{(x-a)^2 + y^2} = \frac{|x+a|}{1}$$

$$\Rightarrow \quad (x-a)^2 + y^2 = (x+a)^2$$

$$\Rightarrow \quad x^2 + a^2 - 2ax + y^2 = x^2 + a^2 + 2ax$$

$$\Rightarrow \quad y^2 = 4ax$$

Hence, the equation of a parabola in the standard form is $y^2 = 4ax$, $a > 0$, with focus $F(a, 0)$ and directrix $x + a = 0$

Sometimes it is called a first standard form or a right hand parabola.

### Some Facts about the Parabola $y^2 = 4ax$, $a > 0$

The equation of the parabola is

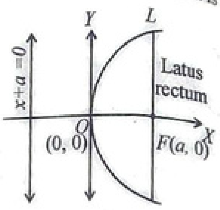
$$F(x, y) = y^2 - 4ax = 0 \qquad \text{....(i)}$$

1. $\because$ $F(x, -y) = (-y)^2 - 4ax = y^2 - 4ax$
$= F(x, y)$

The given parabola is symmetrical about x-axis. This line is the axis of the parabola.

2. Focus is $F(a, 0)$ and directrix is $x + a = 0$.
3. The point 0(0, 0) where the axis of parabola meets the parabola is the vertex of the parabola. It is mid-point of $ZF$ where $F(a, 0)$ is the focus and $Z$ is the foot of perpendicular from focus to the direction.

4. If $x < 0$, then $y^2 = 4ax$ has no real solutions in y and so there is no point on the curve with negative x-coordinate i.e. on the left of y-axis.
5. A chord passing through the focus and perpendicular to the axis of parabola is called latus rectum.
6. **Length of latus-rectum :** Let chord L'L be the latus rectum of the parabola, then L'L passes through focus F($a$, 0) and is perpendicular to x-axis (as shown in fig.)




Let $LF = k$ ($k > 0$), then the points L and L' are ($a$, $k$) and ($a$, $-k$) respectively.

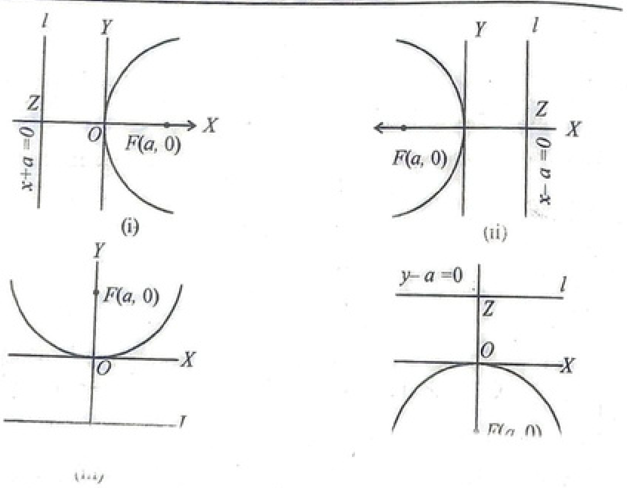
As $L(a, k)$ lies on the parabola $y^2 = 4ax$, we get

$$k^2 = 4a \times a \Rightarrow k = 2a$$

$\therefore$ The points L, L' are ($a$, $2a$), ($a$, $-2a$) and length of latus rectum $= L'L = 2k = 4a$.

The end points of the latus rectum are $L(a, 2a)$. $L'(a, -2a)$ and the equation of the latus rectum is $x - a = 0$

## FOUR STANDARD FORMS OF THE PARABOLA




**Main facts about the parabola**



| Equation | $y^2 = 4ax$ ($a > 0$) Right hand | $y^2 = -4ax$ ($a > 0$) Left hand | $x^2 = 4ay$ ($a > 0$) Upwards | $x^2 = -4ay$ ($a > 0$) Downwards |
|---|---|---|---|---|
| Axis | $y = 0$ | $y = 0$ | $x = 0$ | $x = 0$ |
| Directrix | $x + a = 0$ | $x - a = 0$ | $y + a = 0$ | $y - a = 0$ |
| Focus | ($a$, 0) | ($-a$, 0) | (0, $a$) | (0, $-a$) |
| Vertex | (0, 0) | (0, 0) | (0, 0) | (0, 0) |
| Length of latus-rectum | $4a$ | $4a$ | $4a$ | $4a$ |
| Equation of latus-rectum | $x - a = 0$ | $x + a = 0$ | $y - a = 0$ | $y + a = 0$ |

## ILLUSTRATIVE EXAMPLES

■ **Example 1 : Find the coordinates of focus axis, equation of the directrix and the length of latus-rectum of the parabola represented by the equation $3y^2 = 8x$.**

**Solution :** The given equation is $3y^2 = 8x$

i.e. $$y^2 = \frac{8}{3}x \qquad \text{....(i)}$$

which is the same as $y^2 = 4ax$, So (i) represents a standard (right hand) parabola, and its axis lies along the x-axis. Hence x-axis is the axis of the given parabola.

Also $4a = \frac{8}{3} \Rightarrow a = \frac{2}{3}$, therefore focus is $\left(\frac{2}{3}, 0\right)$ and the equation of directrix is $x + \frac{2}{3} = 0$

i.e. $$3x + 2 = 0$$

$$\text{Length of latus rectum} = 4a = \frac{8}{3}$$

■ **Example 2 : Find the coordinates of focus, equation of directrix and the length of latus-rectum of the parabola represented by the equation $x^2 = -16y$.**

which is comparable with $x^2 = -4ay$, So (i) represents a dourwards parabola.

Hence $4a = 16 \Rightarrow a = 4$

$\therefore$ The focus is (0, $-4$) and the equation of the directrix is

$y - 4 = 0,$

The length of latus-rectum $= 4a = 16.$

■ **Example 3 : Find the equation of the parabola with focus (6, 0) and directrix $x = -6$ also find the length of latus-rectum.**

**Solution :** The focus of the parabola is $F(6, 0)$ and its directrix is the line $x = -6$ i.e. $x + 6 = 0$

Let $P(x, y)$ be any point in the plane of directrix and focus, and MP be the perpendicular distance from P to the directrix then P lies on parabola iff $FP = MP$

$$\Rightarrow \quad \sqrt{(x-6)^2 + (y-0)^2} = \frac{|x+0|}{1}$$

$$\Rightarrow \quad x^2 - 12x + 36 + y^2 = x^2 + 12x + 36$$

$$\Rightarrow \quad y^2 = 24x,$$

which is the required equation of the parabola.

Comparing it with $y^2 = 4ax$, we get $4a = 24$

■ **Example 4 : Find the equation of the parabola with vertex at (0, 0) and focus at (–2, 0).**

**Solution :** Since the focus of the parabola is $F(-2, 0)$ which lies on x-axis, the x-axis is the axis of the parabola.

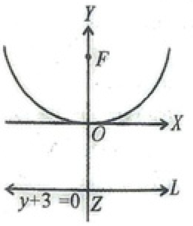
Also, the vertex of the parabola is at $O(0, 0)$ therefore the parabola is

$$y^2 = -4ax, \text{ with } a = 2.$$

Hence the required equation of the parabola is

$$y^2 = -4 \times 2x \text{ i.e. } y^2 = -8x$$

■ **Example 5 : Find the equation of the parabola with vertex at origin and directrix the line $y + 3 = 0$ also find its focus.**




**Solution :** The vertex of the parabola is a origin i.e. the point (0, 0). Let F be the focus of the parabola. The directrix of the parabola is the line $y + 3 = 0$ i.e. $y = -3$.

Therefore, the given parabola is of the third standard form and its equation is $x^2 = 4ay$, where $a = 3$

$\therefore$ The equation of the required parabola is $x^2 = 4 \times 3y$ i.e. $x^2 = 12y$.

The focus of the parabola is $F(0, 3)$



## EXERCISES 6.2

1. Find the length of latus-rectum of each of the following parabolas.
   (i) $y^2 = 8x$ (ii) $y^2 = -12x$ (iii) $x^2 = 16y$
   (iv) $x^2 = -10y$ (v) $3y^2 = -8x$
2. Find the coordinates of the locus of each of the following parabolas.
   (i) $y^2 = 12x$ (ii) $y^2 = -8x$
   (iii) $x^2 = 16y$ (iv) $x^2 = -9y$
3. If the parabola $y^2 = Px$ passes through the point (2, –3) find the length of the latus-rectum.

In each of the following (4 to 7) parabola, find the coordinates of focus, axis of the parabola the equation of the directrix and the length of the latus rectum.

4. (i) $y^2 = 12x$ (ii) $y^2 = 10x$
5. (i) $y^2 = 2\sqrt{3}x$ (ii) $y^2 = -8x$
6. (i) $x^2 = 6y$ (ii) $x^2 = -16y$
7. (i) $x^2 = -9y$ (ii) $3x^2 = 4y$
8. Find the equation of the parabola with
   (i) Focus at (2, 0) and directrix $x = -2$.
   (ii) Focus at (–3, 0) and directrix $y + 3 = 0$
   (iii) Focus at (0, 9) and directrix $y + 4 = 0$
   (iv) Focus at (0, –3) and directrix $y = 3$.
   Also find the length of latus-rectum in each case.
9. If the parabola $y^2 = 4Px$ passes through the point (3, –2) find the length of latus-rectum and the coordinates of the focus.
10. Find the equation of the parabola with vertices at the origin and satisfying the following conditions.
    (i) Focus at (3, 0) (ii) Focus at (0, 2) $x^2 = 4y$
    (iii) Focus at (0, –4) (iv) Directrix $y - 2 = 0$
    (v) Passing through (2, 3) and axis along x-axis.
    (vi) Passing through (5, 2) and symmetric with respect to y-axis.

$x^2 = -4y$

### ANSWERS 6.2

1. (i) 8 (ii) 12 (iii) 16
   (iv) 10 (v) $\frac{8}{3}$

2. (i) $(3, 0)$ (ii) $(-2, 0)$ (iii) $(0, 9)$

(iv) $\left(0, -\dfrac{2}{4}\right)$ 3. $\dfrac{9}{2}$

4. (i) $(3, 0)$; $2x + \sqrt{3} = 0$; $2\sqrt{3}$

(ii) $\left(\dfrac{5}{2}, 0\right)$ x-axis $2x + 5 = 0$; 10

5. (i) $\left(\dfrac{\sqrt{3}}{2}, 0\right)$; x-axis; $2x + \sqrt{3} = 0$; $2\sqrt{3}$

(ii) $(-2, 0)$; $x$-axis $x - 2 = 0$; 8

6. (i) $\left(0, \dfrac{3}{2}\right)$; $y$-axis $2y + 3 = 0$; 6

(ii) $(6, -4)$; $y$-axis; $y - 4 = 0$, 16

7. (i) $\left(0, -\dfrac{9}{4}\right)$, $4y - 9 = 0$, $\dfrac{9}{4}$

(ii) $\left(0, \dfrac{1}{3}\right)$; y-axis, $3y + 1 = 0$; $\dfrac{4}{3}$

8. (i) $y^2 = 8x$; 8 (ii) $y^2 = -12x$, 12

(iii) $x^2 = 16y$; 16 (iv) $x^2 = -12y$; 12

9. $\dfrac{4}{3}, \left(\dfrac{1}{3}, 0\right)$

10. (i) $y^2 = 12x$ (ii) $x^2 = 8y$ (iii) $x^2 = -16y$

(iv) $x^2 = -8y$ (v) $2y^2 = 9x$ (vi) $2x^2 = 25y$

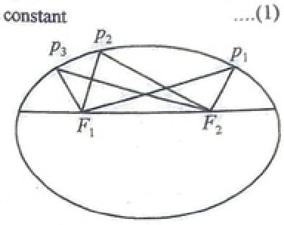
## 6.3 Ellipse

**An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant and is always then the distance between the ellipse. The fixed point (say $F_1$ and $F_2$) are called foci (Plural of focus) of the ellipse.**

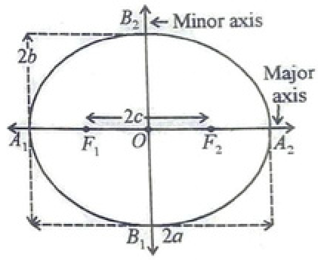
**If $P_1$, $P_2$, $P_3$.....are points on the ellipse than**



$$\begin{aligned} F_1P_1 + F_2P_1 &= F_1P_2 + F_2P_2 \\ &= F_1P_3 + F_2P_3 \\ &= \ldots.. = \text{a constant} \end{aligned} \quad \ldots.(1)$$




The mid-point of the line segment $F_1F_2$ is called the centre of the ellipse. The line passing through the foci of the ellipse is called major axis and the line through the center and perpendicular to the major axis is called minor axis. The points here the ellipse intersects the major axis are called vertices of the ellipse. In fig. $F_1$ and $F_2$ are the two foci of the ellipse, O is the center of the ellipse, the line $F_1F_2$ is the major axis of the ellipse, the points $A_1$ and $A_2$ are vertices at the ellipse the ellipse intersects the minor axis at the points $B_1$ and $B_2$.




The length of the line segment $A_1 A_2$ is called the **length of major axis** and the length of the line segment $B_1 B_2$ is called the **length of minor axis.**

The distance between two foci is by $2c$, the length of major axis by $2a$ and the length of minor axis by $2b$.

Thus $F_1F_2 = 2c$, $A_1A_2 = 2a$ and $B_1B_2 = 2b$.

Therefore, the length of semi major axis is $a$ and the length of semi minor axis is $b$. As O is mid-point of $F_1F_2$, $OF_1 = OF_2 = c$.

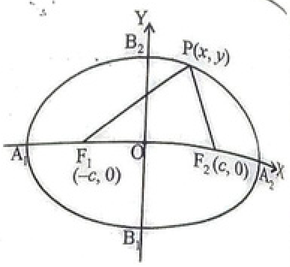
### ECCENTRICITY

The ratio $\dfrac{c}{a}$ is called the eccentricity of the ellipse, it as dentated by e.

Thus, $$e = \frac{c}{a} \text{ i.e. } c = ea.$$

## To find the equation of an ellipse in the standard form $\frac{x^2}{x^2}+\frac{y^2}{b^2}=1$

Let $F_1$, $F_2$ be the two foci and $O$ be the mid-piont of the line segment $F_1F_2$. Take line $F_1F_2$ as $x$-axis with positive direction from $F_1$ to $F_2$. Take O as origin, then the line through O and perpendicular to $F_1F_2$ becomes the $y$-axis (shown in Fig.).




Let $F_1F_2 = 2c$ then $F_1O = OF_2 = c$, so the co-ordinates of $F_1$ and $F_2$ are $(-c, 0)$ and $(c, 0)$ respectively. Let the length of the major axis be $2a$.

Let P$(x, y)$ be any point in plane of the co-ordinate axes, then P lies on *ellipse* if and only if

$$PF_1 + PF_2 = 2a$$

$$\Rightarrow \sqrt{(x+c)^2+(y-0)^2} + \sqrt{(x+c)^2+(y-0)^2} = 2a \quad \text{(using distance formula)}$$

$$\Rightarrow \sqrt{(x+c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$$

$$\Rightarrow (x-c)^2 + y^2 = 4a^2 + (x+c^2) + y^2 - 4a\sqrt{(x+c)^2+y^2} \quad \text{(squaring both sides)}$$

$$\Rightarrow -2\,cx = 4a^2 + 2\,cx - 4a\sqrt{(x+c)^2+y^2}$$

$$\Rightarrow 4a\sqrt{(x+c)^2+y^2} = 4a^2 + 4cx$$

$$\Rightarrow \sqrt{(x+c)^2+y^2} = a + \frac{c}{a}x$$

$$\Rightarrow x^2 + 2cx + c^2 + y^2 = a^2 + \frac{c^2}{a^2}x^2 + 2cx \quad \text{(squaring both sides)}$$

$$\Rightarrow \left(1-\frac{c^2}{a^2}\right)x^2 + y^2 = a^2 - c^2$$

$$\Rightarrow \frac{a^2-c^2}{a^2}x^2 + y^2 = a^2 - c^2 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2-c^2} = 1$$



$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (\because c^2 = a^2 - b^2 \text{ i.e. } b^2 = a^2 - c^2)$$

Hence, the equation of an ellipse in the standard from is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre at origin and major axis along the x-axis.

It is called a *first standard form.*

## Some facts about the ellipse $\frac{x^2}{x^2}+\frac{y^2}{b^2}=1$

The equation of the ellipse is $F(x, y) = \frac{x^2}{x^2}+\frac{y^2}{b^2}-1=0$ .....(1)

We note the following facts about the given ellipse.

1. $F(x, -y) = F(x, y) = F(-x, y)$, also $F(-x, -y) = F(x, y)$

   The ellipse is symmetrical about both the coordinate axes, and about the origin. Thus, the ellipse is symmetrical about both major axis and minor axis.

   As it is symmetrical about origin (0, 0) so origin is the centre of the ellipse.
2. Length of major axis $= A_1A_2 = 2a$ and length of minor axis $= B_1B_2 = 2b$

   The points $A_1(-a, 0)$ and $A_2(a, 0)$ are the vertices of the ellipse.
3. The foci of the ellipse are $F_1(-c, 0)$ and $F_2(c, 0)$, where $c^2 = a^2 - b^2$.
4. Another form of the ellipse is $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$ where $a > b$

**Main facts about the ellipse**

| Equation | $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $a>b>0$ | $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$, $a>b>0$ |
|---|---|---|
| Centre | (0, 0) | (0, 0) |
| Major axis lies along | $x$ + axis | $y$ – axis |
| Length of major axis | $2a$ | $2a$ |
| Length of minor axis | 2b | 2b |
| Foci | (–c, 0), (c, 0) | (0, –c), (0, c) |
| Vertices | $(-a, 0), (a, 0)$ | (0, –a), (0, a) |
| Length of lactus rectum | $\frac{2b^2}{a}$ | $\frac{2b^2}{a}$ |

■ **Example 1 : Find the coordinates of foci, the lengths of major axis, minor axis, latus-rectum and the eccentricity of the ellipse represented by the equation $4x^2 + 9y^2 = 36$**

**Solution :** The equation of the given ellipse is

$$4x^2 + 9y^2 = 36$$

it can be written as $\frac{x^2}{9}+\frac{y^2}{4}=1$ ....(i)

which is comparable with $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, so (i) represents an ellipse of first standard form.

Here $a^2 = 9, b^2 = 4 \Rightarrow a = 3, b = 2.$

we known that $c = \sqrt{a^2-b^2} = \sqrt{9-4} = \sqrt{5}$

$\therefore$ The coordinates of foci are $(-c, 0)$ and $(c, 0)$ i.e. $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$.

The coordinates of vertices are $(-a, 0)$ and $(a, 0)$ i.e. $(-3, 0)$ and $(3, 0)$

Length of major axis $= 2a = 2 \times 3 = 6$

Length of minor axis $= 2b = 2 \times 2 = 4$

Length of latus-rectum $= \frac{2b^2}{a} = \frac{2\times 4}{3} = \frac{8}{3}$

$$\text{Eccentricity} = \frac{c}{a} = \frac{\sqrt{5}}{3}$$

■ **Example 2 : Find the coordinates of foci, the vertices, the lengths of major axis, minor axis, latus-rectum and the eccentricity of the ellipse represented by the equation $4x^2 + y^2 = 100$**

**Solution :** The equation of the given conic is $4x^2 + y^2 = 100$.

it can be written as $\frac{x^2}{25}+\frac{y^2}{100}=1$ .....(i)

Which is comparable with $\frac{x^2}{b^2}+\frac{y^2}{a^2} = 1$, so (i) represent an ellipse of second standard form.

Here $a^2 = 100, b^2 = 25 \Rightarrow a = 10, b = 5.$



we known that $c = \sqrt{a^2-b^2} = \sqrt{100-25} = \sqrt{75} = 5\sqrt{3}$

$\therefore$ The coorinates of foci are $(0, -c)$ and $(0, c)$ i.e. $(0, -5\sqrt{3})$ and $(0, 5\sqrt{3})$.

The coordinates of vertices are $(0, -a)$ and $(0, a)$ i.e. $(0, -10)$ and $(0, 10)$.

Length of major axis $= 2a = 2 \times 10 = 20$

Length o minor axis $= 2b = 2 \times 5 = 10$

Length of latus-rectum $= \frac{2b^2}{a} = \frac{2\times 25}{10} = 5.$

$$\text{Eccentricity} = \frac{c}{a} = \frac{5\sqrt{3}}{10} = \frac{\sqrt{3}}{2}$$

■ **Example 3 : Find the equation of the ellipse whose vertices are (±6, 0) and foci are (±4, 0).**

**Solution :** Here the equation of the ellipse can be taken as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad .....(i)$$

As the vertices are $(\pm 6, 0)$, so $a = 6$ also $c = 4$. We known that $b^2 = a^2 - c^2 = 6^2 - 4^2 = 36 - 16 = 20$ from (i) the equation of the ellipse is $\frac{x^2}{36}+\frac{y^2}{20}=1$

■ **Example 4 : Find the equation of the ellipse whose foci are (0, ±6) and the length of the minor axis is 16.**

**Solution :** Since the foci of the ellipse are $F_1(0, -6)$ and $F_2(0, 6)$ which lie on y-axis.Hence the equation of the ellipse can be taken as.

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1 \quad ....(i)$$

Here $c = 6$

Also, the length of the minor axis is 16

i.e. $2b = 16 \Rightarrow b = 8$

We know that $b^2 = a^2 - c^2 \Rightarrow 8^2 = a^2 - 6^2 \Rightarrow a^2 = 100$

from (i), the equation of the ellipse is $\frac{x^2}{64}+\frac{y^2}{100} = 1.$

■ **Example 5 : Find the equation of the ellipse with centre at origin,**

**major axis on the x axis and passing through the points (4, 3) and (6, 2).**

**Solution :** Since the centre of the ellipse is at origin and major axis along x-axis so its equation can be taken as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{.....(i)}$$

As the ellipse (i) passes through the points (4, 3) and (6, 2)

$$\therefore \quad \frac{16}{a^2}+\frac{9}{b^2}=1 \quad \text{.....(ii)}$$

and

$$\frac{36}{a^2}+\frac{4}{b^2}=1 \quad \text{.....(iii)}$$

Solving eq. (ii) and (iii) we get

$$b^2 = 13 \text{ and } a^2 = 52$$

$\therefore$ The equation of the ellipse is

$$\frac{x^2}{52}+\frac{y^2}{13}=1 \text{ i.e. } x^2+4y^2=52$$

## EXERCISES 6.3

1. Find the length of latus-rectum of each of the following ellipse.

   (i) $\frac{x^2}{16}+\frac{y^2}{9}=1$ (ii) $\frac{x^2}{4}+\frac{y^2}{25}=1$

2. Find the eccentricity of each of the following ellipses.

   (i) $\frac{x^2}{25}+\frac{y^2}{9}=1$ (ii) $3x^2+9y^2=24$

   In each of the ellipses represented by the following (5 to 9) equations find the coordinates of foci the vertices the lengths of major axis, minor axis, latus-rectum and the eccentricity.

3. Find the lengths of major axis and minor axis of each of the following ellipses.

   (i) $4x^2+9y^2=36$ (ii) $4x^2+y^2=100$

4. Find the coordinates of the foci of each of the following ellipses.

   (i) $\frac{x^2}{25}+\frac{y^2}{9}=1$ (ii) $\frac{x^2}{36}+\frac{y^2}{49}=1$



5. (i) $\frac{x^2}{25}+\frac{y^2}{9}=1$ (ii) $\frac{x^2}{36}+\frac{y^2}{16}=1$
6. (i) $9x^2+y^2=36$ (ii) $16x^2+y^2=16$
7. (i) $\frac{x^2}{49}+\frac{y^2}{36}=1$ (ii) $\frac{x^2}{4}+\frac{y^2}{25}=1$
8. (i) $4x^2+y^2=400$ (ii) $9x^2+4y^2=36$
9. (i) $9x^2+16y^2=144$ (ii) $16x^2+9y^2=144$
10. Find the equation of the ellipse whose.

    (i) End points of major axis are (±3, 0) and of minor axis are (0, ± 2)

    (ii) End point of major axis are $(0 \pm \sqrt{5})$ and of minor are (±1, 0).

11. Find the equation of the ellipse whose.

    (i) Vertices are (± 13, 0) and foci are (±5, 0)

    (ii) Vertices are (±5, 0) and foci are ±4, 0)

    (iii) Vertices are (0, ±13) and foci are (0, ±5).

12. Find the equation of ellipse whose:

    (i) Length of major axis is 26 and foci are (±5, 0)

    (ii) Length of major axis is 20 and foci are (0, ±5).

13. Find the equation of the ellipse whose foci are (±3, 0) and length of semi major axis is 9.
14. Find the equation of ellipse whose centre is at origin, foci on x-axis, distance of a focus from centre is 9 and length of semi minor axis is 3.
15. Find the equation of ellipse satisfyins the following conditions :

    (i) Vertices at (0, ±10) e = $\frac{4}{5}$ (ii) Foci at (0, ± 4), e = $\frac{4}{5}$

    (iii) Foci at (±3, 0) passing through (4, 1).

## ANSWERS 6.3

1. (i) $\frac{9}{2}$ (ii) $\frac{8}{5}$
2. (i) $\frac{4}{5}$ (ii) $\frac{1}{2}$

3. (i) 6, 4 (ii) 20; 10

4. (i) $(\sqrt{21},\ 0), (-\sqrt{21},\ 0)$ (ii) $(0,\ \sqrt{13}), (0,\ -\sqrt{13})$

5. (i) $(\pm 4, 0); (\pm 5, 0); 10; 6; \frac{18}{5}, \frac{4}{5}$; (ii) $(\pm 2\sqrt{5}, 0); (\pm 6,\ 0); 12;\ 8;\ \frac{16}{3}; \frac{\sqrt{5}}{3}$

6. (i) $(0, \pm 4\sqrt{2}); (0\ \pm 6); 12; 4; \frac{4}{3}, \frac{2\sqrt{2}}{3}$

   (ii) $(0 \pm \sqrt{15}); (0 \pm 4); 8; 2; \frac{1}{2}; \frac{\sqrt{15}}{4}$

7. (i) $(\pm\sqrt{13}, 0); (\pm 7,\ 0); 14;\ 12;\ \frac{72}{9}; \frac{\sqrt{13}}{7}$

   (ii) $(0, \pm \sqrt{21}); (0, \pm 5); 10; 4; \frac{8}{5}, \frac{\sqrt{21}}{5}$

8. (i) $(0, \pm 10\sqrt{3}); (0, \pm 20); 40; 20; 10; \frac{\sqrt{3}}{2}$; (ii)$(0, \pm\sqrt{5}); (0 \pm 3); 6; 4; \frac{8}{3}; \frac{\sqrt{5}}{3}$

9. (i) $(\pm\sqrt{7},\ 0); (\pm 4,\ 0); 8; 6; \frac{9}{2}; \frac{\sqrt{7}}{4}$

   (ii) $(0,\ \pm\sqrt{7}); (0\ \pm 4); 8; 6;\ \frac{9}{2}; \frac{\sqrt{7}}{4}$

10. (i) $\frac{x^2}{169} + \frac{y^2}{144} = 1$ (ii) $\frac{x^2}{25} + \frac{y^2}{9} = 1$

    (iii) $\frac{x^2}{144} + \frac{y^2}{169} = 1$

11. (i) $\frac{x^2}{9} + \frac{y^2}{4} = 1$ (ii) $\frac{x^2}{1} + \frac{y^2}{5} = 1$

12. (i) $\frac{x^2}{169} + \frac{y^2}{144} = 1$ (ii) $\frac{x^2}{75} + \frac{y^2}{100} = 1$

13. $\frac{x^2}{16} + \frac{y^2}{7} = 1$

14. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

15. (i) $\frac{x^2}{36} + \frac{y^2}{100} = 1$ (ii) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

    (iii) $\frac{x^2}{18} + \frac{y^2}{9} = 1$

# Chapter 7 Quadratic Equations

## 7.1 Quadratic Polynomials

A polynomials of degree 2 is called a quadratic polynomial. The general form of a quadratic polynomial is $ax^2 + bx + c$, where $a$, $b$, $c$ are real number such that $a \neq 0$ and $x$ is a variable.

The real numbers $a$, $b$ and $c$ are known as the coefficients and are also known as the real constants because they are fixed and do not depend on the values of variable x.

Some examples of quadratic polynomials are :

(i) $2x^2 - 3x + 4$ (ii) $x^2 - \sqrt{3x} + 1$

(iii) $x^2 + 9$ (iv) $\sqrt{2}x^2 + x + 3$

(v) $\sqrt{3}x^2 + 2x - \sqrt{7}$.

We will denote as quadratic polynomial

$ax^2 + bx + c$ by $p(x)$ i.e. $p(x) = ax^2 + bx + c$.

### Value of a quadratic polynomial

Let $p(x) = ax^2 + bx + c$ be a quadratic polynomial and let $\alpha$ be a real number. Then, $a\alpha^2 + b\alpha + c$ is known as the value of the quadratic polynomial $p(x)$ and it is denoted by $p(\alpha)$, i.e.,

$$P(\alpha) = a\alpha^2 + b\alpha + c.$$

**Example : Find the value of the quadratic polynomial $P(x) = 2x^2 - 3x + 5$ at $x = -1$.**

**Solution :** We have,

$$P(x) = 2x^2 - 3x + 5.$$

$$\therefore \quad P(-1) = 2 \times (-1)^2 - 3 \times (-1) + 5 = 2 + 3 + 5 = 10$$

## Zero of a quadratic polynomial

A real number $a$ is called a zero of a quadratic polynomial $p(x) = ax^2 + bx + c$, if $p(a) = 0$.

Thus, a zero of a quadratic polynomial is the value of the variable for which the value of the polynomial become zero.

**Example : Show that 2 is a zero of the quadratic polynomial**

$$p(x) = x^2 + x - 6.$$

**Solution :** We have,

$$p(x) = x^2 + x - 6.$$

$\therefore$ $p(2) = 2^2 + 2 - 6 = 4 + 2 - 6 = 0,$

Hence 2 is the zero of $p(x)$

## 7.2 Quadratic Equation

If $p(x)$ is a quadratic polynomial then $p(n) = 0$ is called a quadratic equation.

The general form of a quadratic equation is $ax^2 + bx + c = 0$ where $a, b, c \in R$ and $a \neq 0$.

## Roots of quadratic equation

Let $P(n) = 0$ be a quadratic equation then the zeros of the polynomial p(x) are called the roots of the equation $p(n) = 0$.

Thus, $x = \alpha$ is a root of $p(\alpha) = 0$ if and only if $p(n) = 0$.

■ **Example 1 : Which of the following are quadratic equation ?**

**(i)** $x^2 - 6x + 4 = 0$ **(ii)** $2x^2 - 7x = 0$

**(iii)** $x + \frac{3}{x} = x^2$ **(iv)** $x^2 + \frac{1}{x^2} = 2$

**(v)** $x^2 + 2\sqrt{x} - 3 = 0$ **(vi)** $3x^2 - 4x + 2 = 2x^2 - 2x = 4$

**Solution :** (i) Let $p(x) = x^2 - 6x + 4.$

clearly $p(x) = x^2 - 6x + 4$ is a quadratic polynomial.

Therefore, $x^2 - 6x + 4 = 0$ is a quadratic equation.

(ii) $2x^2 - 7x$ is a quadratic polynomial. So, the given equation is a quadratic equation.

(iii) We have,



$$x + \frac{3}{x} = x^2$$

$$\Rightarrow \quad \frac{x^2 + 3}{x} = x^2$$

$$\Rightarrow \quad x^2 + 3 = x^3 \Rightarrow x^3 - x^2 - 3 = 0$$

clearly, $x^3 - x^2 - 3$, being a polynomial of degre 3, is not a quadratic polynomial. So that given equation is not a quadratic equation.

(iv) We have,

$$x^2 + \frac{1}{x^2} = 2$$

$$\Rightarrow \quad \frac{x^4 + 1}{x^2} = 2$$

Clearly, $x^4 - 2x^2 + 1$ is not a quadratic polynomial. So, the given equation is not a quadratic equation.

(v) Clearly, $x^2 + 2\sqrt{x} - 3$ is not a quadratic polynomial because it contains a term involving $x^{1/2}$ where ½ is not an integer. So, the given equation is not a quadratic equation.

(vi) We have,

$$3x^2 - 4x + 2 = 2x^2 - 2x + 4$$

$$\Rightarrow \quad x^2 - 2x - 2 = 0$$

Clearly, $x^2 - 2x - 2$ is a quadratic polynomial. So, the given equation is a quadratic equation.

■ **Example 2 : If $x = 2$ and $x = 3$ are roots of the quation $3x^2 - 2kx + 2m = 0$, find the values of k and m.**

**Solution :** Since $x = 2$ and $x = 3$ are roots of the equation $3x^2 - 2kx + 2m = 0$

$\therefore$ $3 \times 2^2 - 2k(2) + 2m = 0$ and $3 \times 3^2 - 2k \times 3 + 2m = 0$

$\Rightarrow$ $12 - 4k + 2m = 0$ and $27 - 6k + 2m = 0$

$\Rightarrow$ $12 = 4k - 2m$ and $27 = 6k - 2m$

Solving these two equations, we get

$$k = \frac{15}{2} \text{ and } m = 9.$$

### Exercise 7.1

1. Which of the following are quadratic equations ?

(i) $x^2 + 6x - 4 = 0$ (ii) $\sqrt{3}x^2 - 2x + \frac{1}{2} = 0$

(iii) $x^2 + \frac{1}{x^2} = 5$ (iv) $x - \frac{3}{x} = x^2$

(v) $2x^2 - \sqrt{3x} + 9 = 0$ (vi) $x^2 - 2x - \sqrt{x} - 5 = 0$

2. In each of the following determine whether the given values are solutions of the given equations or not.

of value
exist or not

(i) $x^2 - 3x + 2 = 0; x = 2, x = -1$

(ii) $x^2 - 3\sqrt{3}\,x + 6 = 0; x = \sqrt{3}, x = -2\sqrt{3}$

(iii) $x^2 + x + 1 = 0; x = 0; x = 1$

3. If $x = \frac{2}{3}$ and $x = -3$ are the roots of the equation $ax^2 + 7x + b = 0$, find the value of a and b.

### ANSWERS

1. (i), (ii)
2. (i) $x = 2$ is a solution but $x = -1$ is not a solution.

   (ii) $x = \sqrt{3}$ is a solution but $x = -2\sqrt{3}$ is not a solution.

   (iii) $x = 0$ and $x = 1$ are not solutions
3. $a = 3, b = -6$

## 7.3 SOLVING A QUADRATIC EQUATION BY FACTORIZATION METHOD

Let the quadratic equation be $ax^2 + bx + c = 0;\ a \neq 0$. Let the quadratic polynomial $ax^2 + bx + c$ can be expressed as the product of two linear factors, say $(px + q)$ and $(rx + s)$ where p, q, r, s are real numbers such that $p \neq 0$ and $r \neq 0$. Then,

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad (px + q)(rx + s) = 0$$

$$\Rightarrow \quad px + q = 0 \text{ or, } rx + s = 0$$

Solving these linear equation, we get the possible roots of the given quadratic equation as



$$x = -\frac{q}{p} \text{ and } x = -\frac{s}{r}$$

Following examples will illustrate the above procedure for solving quadratic equations.

### ILLUSTRATIVE EXAMPLES

**Example 1 : Solve the following quadratic equations by factorisation :**

imp **(i) $8x^2 - 22x - 21 = 0$ (iii) $9x^2 - 3x - 2 = 0$**

**(iii) $x^2 + 6x + 5 = 0$**

**Solution :**

(i) We have,

$$8x^2 - 22x - 21 = 0$$

$$\Rightarrow \quad 8x^2 - 28x + 6x - 21 = 0$$

$$\Rightarrow \quad 4x(2x - 7) + 3(2x - 7) = 0$$

$$\Rightarrow \quad (2x - 7)(4x + 3) = 0$$

$$\Rightarrow \quad 2x - 7 = 0 \text{ or } 4x + 3 = 0$$

$$\Rightarrow \quad x = \frac{7}{2} \text{ or } x = -\frac{3}{4}$$

(ii) We have,

$$9x^2 - 3x - 2 = 0$$

$$\Rightarrow \quad 9x^2 - 6x + 3x - 2 = 0$$

$$\Rightarrow \quad 3x(3x - 2) + (3x - 2) = 0$$

$$\Rightarrow \quad (3x - 2) + 1(3x + 1) = 0$$

$$\Rightarrow \quad 3x - 2 = 0 \text{ or } 3x + 1 = 0$$

$$\Rightarrow \quad x = \frac{2}{3} \text{ or } x = -\frac{1}{3}$$

(iii) we have

$$x^2 + 6x + 5 = 0$$

$$\Rightarrow \quad x^2 + 5x + x + 5 = 0$$

$$\Rightarrow \quad x(x + 5) + 1(x + 5) = 0$$

$$\Rightarrow \quad (x + 5)(x + 1) = 0$$

$$\Rightarrow \quad x + 5 = 0 \text{ or } x + 1 = 0$$

$$\Rightarrow \quad x = -5 \text{ or } x = -1$$

**Example 2 : Solve the following quadratic equations by factorisation method.**

(i) $x^2 + 2\sqrt{2}x - 6 = 0$ (ii) $\sqrt{3}x^2 + 10x + 7\sqrt{3} = 0$

**Solution :** We have,

$$x^2 + 2\sqrt{2}x - 6 = 0$$

$$\Rightarrow x^2 + 3\sqrt{2}x - \sqrt{2}x - 6 = 0$$

$$\Rightarrow x(x + 3\sqrt{2}) - \sqrt{2}(x + 3\sqrt{2}) = 0$$

$$\Rightarrow (x + 3\sqrt{2})(x - \sqrt{2}) = 0$$

$$\Rightarrow x + 3\sqrt{2} = 0 \text{ or } x - \sqrt{2} = 0$$

$$\Rightarrow x = -3\sqrt{2} \text{ or } x = \sqrt{2}$$

(ii) We have,

$$\sqrt{3}x^2 + 10x + 7\sqrt{3} = 0$$

$$\Rightarrow \sqrt{3}x^2 + 3x + 7x + 7\sqrt{3} = 0$$

$$\Rightarrow \sqrt{3}x(x + \sqrt{3}) + 7(x + \sqrt{3}) = 0$$

$$\Rightarrow (x + \sqrt{3})(\sqrt{3}x + 7) = 0$$

$$\Rightarrow x + \sqrt{3} = 0 \text{ or } \sqrt{3}x + 7 = 0$$

$$\Rightarrow x = -\sqrt{3} \text{ or } x = -7/\sqrt{3}$$

**Example 3 : Solve the following quadratic equations by factorization method.**

(i) $x^2 - 9 = 0$ (ii) $x^2 - 8x + 16 = 0$

**Solution :** (i) We have,

$$x^2 - 9 = 0$$

$$\Rightarrow (x - 3)(x + 3) = 0$$

$$\Rightarrow x - 3 = 0 \text{ or } x + 3 = 0$$

$$\Rightarrow x = 3 \text{ or } x = -3$$

(ii) We have,

$$x^2 - 8x + 16 = 0$$

$$\Rightarrow (x - 4)^2 = 0$$



$$\Rightarrow x = 4, x = 4$$

## EXERCISES 7.2

Solve the following quadratic equation by factorization :

1. $4x^2 + 5x = 0$
2. $9x^2 - 3x - 2 = 0$
3. $6x^2 - x - 2 = 0$
4. $6x^2 + 11x + 3 = 0$
5. $5x^2 - 3x - 2 = 0$
6. $48x^2 - 13x - 1 = 0$
7. $3x^2 + 11x + 10 = 0$
8. $25x(x + 1) = -4$
9. $10x - \frac{1}{x} = 3$
10. $\frac{2}{x^2} - \frac{5}{x} + 2 = 0$ 2
11. $4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 0$

## ANSWERS 7.2

| | | |
|---|---|---|
| 1. $0, -\frac{5}{4}$ | 2. $\frac{2}{3}, -\frac{1}{3}$ | 3. $\frac{2}{3}, -\frac{1}{2}$ |
| 4. $-\frac{3}{2}, -\frac{1}{3}$ | 5. $-\frac{2}{5}, 1$ | 6. $-\frac{1}{16}, \frac{1}{3}$ |
| 7. $-\frac{5}{3}, -2$ | 8. $-\frac{4}{5}, -\frac{1}{5}$ | 9. $\frac{1}{2}, -\frac{1}{5}$ |
| 10. $2, \frac{1}{2}$ | 11. $\frac{\sqrt{3}}{4}, -\frac{2}{\sqrt{3}}$ | |

## 7.4 Solving a Quadratic Equation by the Method of Completion of Squares (Shridharacharya's Rule)

Consider the quadratic equation

$$ax^2 + bx + c = 0, a \neq 0 \quad \text{.....(i)}$$

$$\Rightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \quad \text{[Dividing throughout by } a\text{]}$$

$$\Rightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

[Adding $\left(\frac{b}{2a}\right)^2$ i.e. $(\frac{1}{2}$ coeff of $x)^2$ on both sides]

$$\Rightarrow \quad x^2+2\left(\frac{b}{2a}\right)x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x+\frac{b}{2a}\right)^2=\left(\frac{b^2-4ac}{4a^2}\right)$$

$$\Rightarrow \quad x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a} \text{ [Taking square root of both sides)}$$

$$\Rightarrow \quad x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=-\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Rightarrow \quad x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ or, } x=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Thus, if $b^2-4ac\geq 0$ then the quadratic equation $ax^2+bx+c=0$ has two roots $\alpha$ and $\beta$, given by

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

The formula given in equation (1) is known as Shri Dharacharya's rule

## NATURE OF THE ROOTS OF A QUADRATIC EQUATION

In the above discussion, we have seen that the roots of the quadratic equation $ax^2+bx+c=0$, $a\neq 0$ are real if $b^2-4ac\geq 0$. Thus, if $b^2-4ac<0$, the equation will have no real roots. Further more, i $b^2-4ac=0$ then the roots are equal or coincident and each being equal to $-\frac{b}{2a}$.

**Discriminant :** If $ax^2+bx+c=0$, $a\neq 0$ is a quadratic equation, then the expression $b^2-4ac$ is known as its discriminant and is denoted by D.



(i) If $D>0$ then there are two real and distinct roots, given by

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}=\frac{-b+\sqrt{D}}{2a}$$

$$\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}=\frac{-b-\sqrt{D}}{2a}$$

(ii) If $D=0$, then $\alpha=\frac{-b}{2a}=\beta$ i.e. roots are equal each being equal to $\frac{-b}{2a}$.

(iii) If $D<0$, then there are no real roots.

## ILLUSTRATIVE EXAMPLES

**Example 1 : Write the discriminant of the following quadratic equations.**

**(i) $x^2-4x+2=0$ (ii) $3x^2+2x-1=0$**

**(iii) $x^2-4x+a=0$ (iv) $\sqrt{3}x^2-2\sqrt{2}x-2\sqrt{3}=0$**

**(v) $x^2+x+1=0$**

**Solution :** (i) The given equation is

$$x^2-4x+2=0$$

Here, $a=1, b=-4$ and $c=2$

$\therefore$ $D=b^2-4ac=(-4)^2-4\times 1\times 2=16-8=8$

(ii) The given equation is

$$3x^2+2x-1=0$$

Here $a=3, b=2$ and $c=-1$

$D=b^2-4ac=2^2-4\times 3\times -1=4+12=16$

(iii) The given equation is

$$x^2-4x+a=0$$

Here $a=1, b=-4$ and $c=a$.

$\therefore$ $D=b^2-4ac=(-4)^2-4\times 1\times a=16-4a$

(iv) The given equation is

$$\sqrt{3}x^2-2\sqrt{2}x-2\sqrt{3}=0$$

Here $a=\sqrt{3}, b=-2\sqrt{2}$ and $C=-2\sqrt{3}$

$\therefore$ $D=b^2-4ac=(-2\sqrt{2})^2-4\times\sqrt{3}\times -2\sqrt{3}$

$=8+24=32$

$D = b^2 - 4ac$

III case $\Rightarrow$

$D > 0$ real and distinct
$D = 0$ real and equal
$D < 0$ not real



(v) The given equation is

$$x^2 + x + 1 = 0$$

Here $a = 1, b = 1$ and $c = 1$

$\therefore$ $D = b^2 - 4ac$

$= 1^2 - 4 \times 1 \times 1 = -3$

■ **Example 2 : Determine the nature of the roots of the following quadratic equations :**

**(i)** $2x^2 + x - 1 = 0$ **(ii)** $x^2 - 4x + 4 = 0$

**(iii)** $x^2 + x + 1 = 0$ **(iv)** $4x^2 - 4x + 1 = 0$

**(v)** $2x^2 + 5x + 5 = 0$

**Solution :** (i) The given quadratic equation is

$$2x^2 + x - 1 = 0$$

Here $a = 2, b = 1,$ and $c = -1$

$\therefore$ $D = b^2 - 4ac = 1^2 - 4 \times 2 \times -1 = 9$

Since $D > 0$, therefore roots of the given equation are real and distinct.

(ii) The given equation is

$$x^2 - 4x + 4 = 0$$

Here, $a = 1, b = -4$ and $c = 4.$

$\therefore$ $D = b^2 - 4ac = (-4)^2 - 4 \times 1 \times 4 = 0$

Since D = 0, therefore roots of the given equation are real and equal.

(iii) The given equation is

$$x^2 + x + 1 = 0$$

Here $a = 1, b = 1$ and $c = 1$

$\therefore$ $D = b^2 - 4ac = 1^2 - 4 \times 1 \times 1 = -3$

Since D < 0, therefore roots of the given equation are not real.

(iv) The given equation is

$$4x^2 - 4x + 1 = 0$$

Here, $a = 4, b = -4$ and $c = 1$

$\therefore$ $D = b^2 - 4ac = (-4)^2 - 4 \times 4 \times 1 = 0$

Since D = 0; therefore roots of the given equation are real and equal.

(v) The given equation is

$$2x^2 + 5x + 5 = 0$$

Here $a = 2, b = 5$ and $c = 5$

First two
right two time

Ex. Imp.



$$D = b^2 - 4ac = 5^2 - 4 \times 2 \times 5 = 25 - 40 = -15$$

Since D < 0 therefore roots of the given equation are not real.

■ **Example 3 : In the following determine whether the given quadratic equations have real roots and if so find the roots.**

**(i)** $9x^2 + 7x - 2 = 0$ **(ii)** $2x^2 + 5\sqrt{3}x + 6 = 0$

**(iii)** $3x^2 + 2\sqrt{5}x - 5 = 0$ **(iv)** $x^2 + 5x + 5 = 0$

**(v)** $25x^2 + 20x + 7 = 0$

**Solution :** (i) The given equation is $9x^2 + 7x - 2 = 0$

Here, $a = 9, b = 7$ and $c = -2$

$\therefore$ $D = b^2 - 4ac = 7^2 - 4 \times 9 \times -2 = 49 + 72 = 121 > 0$

So the given equation has real roots, given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{-7 + \sqrt{121}}{2 \times 9} = \frac{-7 + 11}{18} = \frac{4}{18} = \frac{2}{9}$$

and $$\beta = \frac{-b - \sqrt{D}}{2a} = \frac{-7 - \sqrt{121}}{2 \times 9} = \frac{-7 - 11}{18} = -1$$

(ii) The given equation is

$$2x^2 + 5\sqrt{3}x + 6 = 0$$

Here, $a = 2, b = 5\sqrt{3}$ and $c = 6.$

$\therefore$ $D = b^2 - 4ac = 75 - 4 \times 2 \times 6 = 27 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b + \sqrt{D}}{2a} = \frac{-5\sqrt{3} + \sqrt{27}}{2 \times 2}$$

$$= \frac{-5\sqrt{3} + 3\sqrt{3}}{4} = \frac{-2\sqrt{3}}{4} = \frac{-\sqrt{3}}{2}$$

and $$\beta = \frac{-b - \sqrt{D}}{2a} = \frac{-5\sqrt{3} - \sqrt{27}}{2 \times 2}$$

$$= \frac{-5\sqrt{3} - 3\sqrt{3}}{4} = -2\sqrt{3}$$

(iii) The given equation is

$$3x^2 + 2\sqrt{5}\,x - 5 = 0$$

Here $a = 3, b = 2\sqrt{5}$ and $c = -5$

$\therefore$ $D = b^2 - 4ac = (2\sqrt{5})^2 - 4 \times 3 \times -5 = 20 + 60 = 80 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-b+\sqrt{D}}{2a} = \frac{-2\sqrt{5}+\sqrt{80}}{2\times 3}$$

$$= \frac{-2\sqrt{5}+4\sqrt{5}}{6} = \frac{2\sqrt{5}}{6} = \frac{\sqrt{5}}{3}$$

and $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-2\sqrt{5}-\sqrt{80}}{2\times 3}$$

$$= \frac{-2\sqrt{5}-4\sqrt{5}}{6} = -\sqrt{5}$$

(iv) The given equation is

$$x^2 + 5x + 5 = 0$$

Here, $a = 1, b = 5$ and $c = 5$

$\therefore$ $D = b^2 - 4ac = 25 - 4 \times 1 \times 5 = 5 > 0$

So, the given equation has real roots, given by

$$\alpha = \frac{-6+\sqrt{D}}{2a} = \frac{-5+\sqrt{5}}{2}$$

and $$\beta = \frac{-b-\sqrt{D}}{2a} = \frac{-5-\sqrt{5}}{2}$$

(v) The given equation is

$$25x^2 + 20x + 7 = 0$$

Here, $a = 25, b = 20$ and $c = 7$

$\therefore$ $D = b^2 - 4ac = (20)^2 - 4 \times 25 \times 7 = 400 - 700 = -300 < 0$

So, the given equation has no real roots.



## EXERCISES 7.3

$D = b^2 - 4aC$

1. Write the discriminant of the following quadratic equations. only Answ
   (i) $2x^2 - 5x + 3 = 0$ (ii) $x^2 + 2x + 4 = 0$
   (iii) $(x-1)(2x-1)\ 0$ (iv) $\sqrt{3}x^2 + 2\sqrt{2}\,x - 2\sqrt{3} = 0$

2. Determine the nature of the roots of the following quadratic equations.
   (i) $2x^2 - 3x + 4 = 0$ (ii) $\frac{3}{5}x^2 - \frac{2}{3}x + 1 = 0$ $D = b^2 - 4ac$
   (iii) $x^2 + 2\sqrt{3}\,x - 1 = 0$ (iv) $3x^2 - 2\sqrt{6}\,x + 2 = 0$ and types

3. In the following determine whether the given quadratic equations have real roots and if so find the roots.
   (i) $x^2 + x + 2 = 0$ (ii) $\sqrt{3}x^2 + 10x - 8\sqrt{3} = 0$ roots value find
   (iii) $3x^2 - 2x + 2 = 0$ (iv) $2x^2 - 2\sqrt{6}\,x + 3 = 0$
   (v) $3x^2 + 2\sqrt{5}\,x - 5 = 0$ (vi) $x^2 - 2x + 1 = 0$

4. Find the values of k for which the roots are and equal in each of the following equations.
   (i) $kx^2 + 4x + 1 = 0$ (ii) $kx^2 - 2\sqrt{5}\,x + 4 = 0$
   (iii) $3x^2 - 5x + 2k = 0$ (iv) $4x^2 + kx + 9 = 0$
   (v) $2kx^2 - 40x + 25 = 0$

## ANSWERS 7.3

1. (i) 1 (ii) $-12$ (iii) 1 (iv) 32
2. (i) Not real
   (iii) Not real (iv) Real and distinct
   (v) Real and equal
3. (ii) $\frac{2\sqrt{3}}{3}, -4\sqrt{3}$ (iv) $\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}$
   (v) $\frac{\sqrt{5}}{3}, -\sqrt{5}$ (vi) 1, 1
4. (i) 4 (ii) $\frac{5}{4}$
   (iii) $\frac{24}{25}$ (iv) $\pm 6$ (v) 8

## 7.5 Solutions of Problems Involving Quadratic Equations

In this section, we will discuss some simple problems on practical applications of quadratic equations.

■ **Example 1 : The sum of the squares of two consecutive natural numbers is 313. Find the numbers.**

$x, (x+1)$

**Solution :** Let two consecutive natural numbers be $x$ and $x + 1$

Then, $x^2 + (x+1)^2 = 313$

$\Rightarrow$ $2x^2 + 2x + 1 = 313$

$\Rightarrow$ $2x^2 + 2x - 312 = 0$

$\Rightarrow$ $x^2 + x - 156 = 0$

$\Rightarrow$ $x^2 + 13x - 12x - 156 = 0$

$\Rightarrow$ $x(x+13) - 12(x+13) = 0$

$\Rightarrow$ $(x+13)(x-12) = 0$

$\Rightarrow$ $x + 13 = 0$ or $x - 12 = 0$

$\Rightarrow$ $x = 12$ or $x = -13$

Hence, the two consecutive natural numbers are 12 and 13.

■ **Example 2 : Divide 16 into two parts such that twice the square of the larger part exceeds the square of the smaller part by 164.**

**Solution :** Let the larger part be $x$. Then the smaller part $= 16 - x$.

According to question

$$2x^2 = (16-x)^2 + 164$$

$\Rightarrow$ $2x^2 - (16-x)^2 - 164 = 0$

$\Rightarrow$ $x^2 + 32x - 420 = 0$

$\Rightarrow$ $(x+42)(x-10) = 0$

$\Rightarrow$ $x = -42$ or $x = 10$

$\Rightarrow$ $x = 10$

Hence, the required parts are 10 and 6.

■ **Example 3 : The sum of two numbers is 15. If the sum of their reciprocals is $\frac{3}{10}$ then find the numbers.**

**Solution :** Let the required numbers be $x$ and $15-x$. Then



$$\frac{1}{x} + \frac{1}{15-x} = \frac{3}{10}$$

$\Rightarrow$ $\frac{15-x+x}{x(15-x)} = \frac{3}{10}$

$\Rightarrow$ $\frac{15}{x(15-x)} = \frac{3}{10}$

$\Rightarrow$ $150 = 3x(15-x)$

$\Rightarrow$ $150 = 45x - 3x^2$

$\Rightarrow$ $x^2 - 15x + 50 = 0$

$\Rightarrow$ $x^2 - 10x - 5x - 50 = 0$

$\Rightarrow$ $x(x-10) - 5(x-10) = 0$

$\Rightarrow$ $(x-10)(x-5) = 0$

$\Rightarrow$ $x - 10 = 0$ or $x - 5 = 0$

$\Rightarrow$ $x = 10$ or, $x = 5$

Hence, the two numbers are 10 and 5.

■ **Example 4 : A fast train takes 3 hours less than a slow train for a journey of 600 km. If the speed of the slow train is 10 km/hr less than that of the fast train, find the speeds of two trains.**

**Solution :** Let the speed of slow train be $x$ km/hr. Then, speed of the fast train is $(x+10)$ km/hr.

Time taken by the slow train to cover 600 km $= \frac{600}{x}$ hrs.

Time taken by the fast train to cover 600 km $= \frac{600}{x+10}$ hrs.

$\therefore$ $\frac{600}{x} - \frac{600}{x+10} = 3$

$\Rightarrow$ $\frac{600(x+10) - 600x}{x(x+10)} = 3$

$\Rightarrow$ $\frac{6000}{x^2 + 10x} = 3$

$\Rightarrow$ $3(x^2 + 10x) = 6000$

$\Rightarrow$ $x^2 + 10x - 2000 = 0$

$\Rightarrow \quad x^2 + 50x - 40x - 2000 = 0$

$\Rightarrow \quad x(x + 50) - 40(x + 50) = 0$

$\Rightarrow \quad (x + 50)(x - 40) = 0$

$\Rightarrow \quad x = -50 \text{ or } x = 40 \Rightarrow x = 40$ [$\because$ speed cannot be negative]

Hence, the speeds of two trains are 40 km/hr and 50km/hr.

■ **Example 5 : A two digit number is such that the product of the digits is 14. When 45 is added to the number then the digits are reversed. Find the number.**

**Solution :** Let the tens digit be $x$. Then, unit digit = $\frac{14}{x}$

$\therefore \quad \text{Number} = 10x + \frac{14}{x}$

and number formed by reversing the digits = $10 \times \frac{14}{x} + x$

$\therefore \quad 10x + \frac{14}{x} + 45 = 10 \times \frac{14}{x} + x$ [Given]

$\Rightarrow \quad 10x + \frac{14}{x} + 45 = \frac{140}{x} + x$

$\Rightarrow \quad 9x - \frac{126}{x} + 45 = 0$

$\Rightarrow \quad 9x^2 + 45x - 126 = 0$

$\Rightarrow \quad x^2 + 5x - 14 = 0$

$\Rightarrow \quad x^2 + 7x - 2x - 14 = 0$

$\Rightarrow \quad (x + 7)(x - 2) = 0$

$\Rightarrow \quad x = -7 \text{ or } x = 2$

$\Rightarrow \quad x = 2$ [$\because$ Digits cannot be negative]

Hence, the required number = $10 \times 2 + \frac{14}{2} = 27$

■ **Example 6 : A train travels a distance of 300 km at a constant speed. If the speed of the train is increased by 5 km per hour, the journey would have taken 2 hours less. Find the original speed of the train.**



**Solution :** Let $x$ km/ hr be the constant speed of the train then, time taken to cover 300 km = $\frac{300}{x}$ hrs. Time taken to cover 300 km when the speed is increased by 5 km/hr = $\frac{300}{x+5}$ hours.

According to question

$\therefore \quad \frac{300}{x} - \frac{300}{x+5} = 2$

$\Rightarrow \quad \frac{300(x+5) - 300x}{x(x+5)} = 2$

$\Rightarrow \quad \frac{300x + 1500 - 300x}{x^2 + 5x} = 2$

$\Rightarrow \quad 2x^2 + 10x = 1500$

$\Rightarrow \quad x^2 + 5x - 750 = 0$

$\Rightarrow \quad x^2 + 30x - 25x - 750 = 0$

$\Rightarrow \quad x(x + 30) - 25(x + 30) = 0$

$\Rightarrow \quad (x - 25)(x + 30) = 0$

$\Rightarrow \quad x = 25 \text{ or } x = -30$

$\because$ Speed can not be negative so $x = -30$ is rejected.

Hence the original speed of the train is 25km/hr.

■ **Example 7 : The hypotenuse of a right angled triangle is 6 metres more than twice the shortest side. If the third side is 2 metres less than the hypotenuse find the sides of the triangle.**

**Solution :** Let the length of the shortest side be $x$ metres. Then,

Hypotenuse = $(2x + 6)$ metres

And, the third side = $(2x + 6 - 2)$ metres = $(2x + 4)$ metres. By Pythagoras theorem, we have

$(2x + 6)^2 = x^2 + (2x + 4)^2$

$\Rightarrow \quad x^2 - 8x - 20 = 0$

$\Rightarrow \quad x^2 - 10x + 2x - 20 = 0$

$\Rightarrow \quad (x - 10)(x + 2) = 0$

$\Rightarrow \quad x = 10 \text{ or } x = -2$

$\Rightarrow \quad x = 10$ [Since side of a triangle is never negative]

∴ Length of the shortest side = 10 metres.

Length of the hypotenuse = $(2x + 6)$ metres = 26 metres

Length of the third side = $(2x + 4)$ metres = 24 metres

Hence, the sides of the triangle are 10m, 26m and 24m.

**■ Example 8 : One year ago, a man was 8 times as old as his son. Now his age is equal to the square of his son's age. Find their present ages.**

**Solution :** Suppose, one year ago, son's age be x years.

Then, man's age one year ago 8 = $8x$ years.

∴ Present age of son = $(x + 1)$ years and present age of man = $(8x + 1)$ years.

$$\therefore \quad 8x + 1 = (x + 1)^2 \qquad \text{[Given]}$$
$$\Rightarrow \quad x^2 - 6x = 0$$
$$\Rightarrow \quad x(x - 6) = 0$$
$$\Rightarrow \quad x = 0 \text{ or } x = 6$$
$$\Rightarrow \quad x = 6 \qquad [\because \text{ age can not be 0 year}]$$

So, present age of son = $(x + 1)$ years = 7 years and present age of man = $(8x + 1)$ years = 49 years.

**■ Example 9 : The product of Kamal's age five years ago with his age 9 years later is 15. Find Kamal's present age.**

**Solution :** Let Kamal's present age be $x$ year then,

His age 5 years ago = $(x - 5)$ years.

His age 9 year later = $(x + 9)$ years.

$$\therefore \quad (x - 5)(x + 9) = 15 \qquad \text{[Given]}$$
$$\Rightarrow \quad x^2 + 9x - 60 = 0$$
$$\Rightarrow \quad x^2 + 10x - 6x - 60 = 0$$
$$\Rightarrow \quad (x + 10)(x - 6) = 0$$
$$\Rightarrow \quad x = 6 \text{ or } x = -10$$
$$\text{But} \quad x \neq -10 \text{ so, } x = 6$$

Hence, Kamal's present age is 6 years.

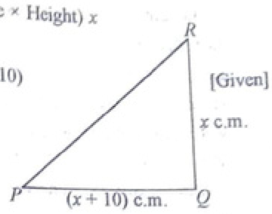
**■ Example 10 : The area of a right angled triangle is 600 $cm^2$. If the base of the triangle exceeds the altitude by 10 cm, find the dimensions of the triangle.**

**Solution :** Let the altitude QR of right angled triangle PQR be $x$ cm. Then,

Base = PQ = $(x + 10)$ cm.



$$\therefore \quad \text{Area} = \frac{1}{2}(\text{Base} \times \text{Height})\, x$$
$$\Rightarrow \quad 600 = \frac{1}{2}(x + 10) \qquad \text{[Given]}$$
$$\Rightarrow \quad x^2 + 10x = 1200$$
$$\Rightarrow \quad x^2 + 10x - 1200 = 0$$
$$\Rightarrow \quad x^2 + 40x - 30x - 1200 = 0$$
$$\Rightarrow \quad x(x + 40) - 30(x + 40) = 0$$
$$\Rightarrow \quad (x + 40)(x - 30) = 0$$




$$x = -40, 30$$
$$\therefore \quad x = 30 \qquad [\because \text{ side can't be negative}]$$
$$\therefore \quad PQ = 40 \text{ cm}, QR = 30 \text{ cm}, PQ = 50$$

## EXERCISES 7.4

1. Find two consecutive numbers whose squares have the sum 85.
2. The sum of two numbers is 98 and their product is 432. Find the numbers.
3. Two squares have sides x cm and (x +4) cm. The sum of their areas is 656 $cm^2$. Find the sides of the squares.
4. Find two consecutive natural numbers whose product is 20.
5. A piece of cloth costs Rs 35. If the piece were 4m longer and each metre costs Rs one less,the cost would remain unchanged. How long is the piece ?
6. A two digit number is such that the product of the digits is 16. When 54 is subtracted from the number the digits are interchanged. Find the number.
7. A passenger train takes 3 hours less for a journey of 360 km. If its speed is increased by 10 km/hr from its usual speed. What is the usual speed ?
8. The sum of the ages of a man and his son is 45 years. Five years ago the product of their ages was four times the man's age at that time. Find their present ages.
9. The product of Naresh's age five years ago and his age nine years later is 15. Determine his present age.
10. The product of Mona's age five years ago and her age 8 years later is 30. Find her present age.
11. Two numbers differ by 3 and their product is 504, find the numbers.

12. Rs 9000 were divided equally among a certain number of persons. Had there been 20 more persons, each would have get Rs 160 less. Find the original number of persons.

**ANSWERS**

| | |
|---|---|
| 1. 6, 7 or – 6, – 7 | 2. 36, 12 |
| 3. 16 cm, 20 cm | 4. 4, 5 |
| 5. 10 m | 6. 82 |
| 7. 30 km/hr | 8. 36 years, 9 years |
| 9. 6 years, | 10. 7 years |
| 11. 21, 24, or – 24, – 21 | 12. 25 |

## 7.6 RELATION BETWEEN THE ROOTS OF A QUADRATIC EQUATION

Let roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, $b, c \in R$ be $\alpha$ and $\beta$, then we know that

1. When $b^2 - 4ac > 0$, then roots are

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

2. When $b^2 - 4ac = 0$, then roots are $\alpha = -\frac{b}{2a} = \beta$.

3. When $b^2 - 4ac < 0$, then root are imaginary

$$\alpha = \frac{-b+i\sqrt{4ac-b^2}}{2a} \text{ and } \beta = \frac{-b-i\sqrt{4ac-b^2}}{2a}$$

Now we have to find the sum and the product of roots in each case.

**Case 1.** $$\alpha+\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}$$

and $$\alpha\beta = \frac{\left(-b+\sqrt{b^2-4ac}\right)}{2a}\frac{\left(-b-\sqrt{b^2-4ac}\right)}{2a}$$



$$= \left(\frac{b-\sqrt{b^2-4ac}}{2a}\right)\left(\frac{b+\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{b^2-\left(b^2-4ac\right)}{2a} = \frac{4ac}{2a} = \frac{c}{a}$$

**Case 2.** $$\alpha+\beta = \left(-\frac{b}{2a}\right)+\left(-\frac{b}{2a}\right) = -\frac{b}{a}$$

and $$\alpha\beta = \left(-\frac{b}{2a}\right)\left(-\frac{b}{2a}\right) = \frac{b^2}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

$[\because b^2 = 4ac]$

Hence from the above cases, we can find that

$$\text{Sum of roots} = \alpha+\beta = -\frac{b}{a} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{and product of roots } \alpha\beta = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

## 7.7 FORMATION OF A QUADRATIC EQUATION FROM GIVEN ROOTS

We have seen in previous section that if $\alpha$ and $\beta$ are roots of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$ then

$$\alpha+\beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a} \quad .....(1)$$

but $ax^2 + bx + c = 0$

or $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ [dividing by $a$]

or $x^2 - (a + b)x + ab = 0$ [by (1)]

which is the required quadratic equation in $x$ having roots $\alpha$ and $\beta$.

Hence required quadratic equation in terms of sum and product of roots is :

$$x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$$

## 7.8 Values of symmetric expressions of roots

Let $\alpha$ and $\beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$. If in any expression in $\alpha$ and $\beta$, if $\alpha$ is replaced by $\beta$ and $\beta$ is replaced by $\alpha$ and the expression remains unchanged, then such an expression is known as a symmetric expression. To find the value of a symmetric expression, first we change this expression in terms of $\alpha + \beta$ and $\alpha\beta$, then from the given equation we substitute values of $\alpha + \beta$ and $\alpha\beta$. For this the following transformations are very useful :

(i) $a - b = \sqrt{(a+b)^2 - 4ab}$

(ii) $a^2 + b^2 = (a + b)^2 - 2ab$

(iii) $a^2 - b^2 = (a + b)(a - b) = (a + b)\sqrt{(a+b)^2 - 4ab}$

(iv) $a^3 + b^3 = (a + b)^3 - 3ab\,(a + b)$

(v) $a^3 - b^3 = (a - b)\,(a^2 + ab + b^2)$
$= (a - b)\,\{(a + b)^2 - ab\}$
$= \sqrt{(a+b)^2 - 4ab}\,\{(a + b)^2 - ab\}$

(vi) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$
$= \{(a + b)^2 - 2ab\}^2 - 2(\alpha\beta)^2$

(vii) $\alpha^4 - \beta^4 = (\alpha^2 + \beta^2)(\alpha^2 - \beta^2)$
$= (\alpha + \beta)(\alpha^2 + \beta^2)(\alpha - \beta)$
$= (\alpha + \beta)\,\{(\alpha + \beta)^2 - 2\alpha\beta\}\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}$

### Illustrative Examples

■ **Example 1. Find conjugate root and quadratic equation whose one root is $3 - \sqrt{2}$.**

**Solution :** Since one given root is $3 - \sqrt{2}$, so the other conjugate root will be $3 + \sqrt{2}$.

Now, sum of roots $= (3 - \sqrt{2}) + (3 + \sqrt{2}) = 6$

and product of roots $= (3 - \sqrt{2})(3 + \sqrt{2}) = 9 - 2 = 7$

Hence required equation is :

$x^2$ – (sum of roots) $x$ + (product of roots) = 0

or $x^2 - 6x + 7 = 0$

■ **Example 2. If $\alpha$ and $\beta$ are roots of the equation $2x^2 - 3x - 6 = 0$ then find the quadratic equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$. [R.U. 2015]**



**Solution :** Since, roots of the quadratic equation $2x^2 - 3x - 6 = 0$ are $\alpha$ and $\beta$, therefore,

$\therefore \quad \alpha + \beta = \frac{3}{2}$ and $\alpha\beta = -3$

Now $(\alpha^2 + 2) + (\beta^2 + 2) = (\alpha^2 + \beta^2) + 4$
$= (\alpha + \beta)^2 - 2\alpha\beta + 4$

$$= \left(\frac{3}{2}\right)^2 - 2(-3) + 4 = \frac{9}{4} + 10 = \frac{49}{4}$$

and $(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4$
$= \alpha^2\beta^2 + 2\{(\alpha + \beta)^2 - 2\alpha\beta\} + 4$

$$= (-3)^2 + 2\left\{\left(\frac{3}{2}\right)^2 - 2(-3)\right\} + 4$$

$$= 9 + 2\left(\frac{9}{4} + 6\right) + 4 = 25 + \frac{9}{2} = \frac{59}{2}$$

Hence the required equation is

$x^2$ – (sum of roots) $x$ + (product of roots) = 0

$$\Rightarrow \quad x^2 - \frac{49}{4}x + \frac{59}{2} = 0$$

$$\Rightarrow \quad 4x^2 - 49x + 118 = 0$$

■ **Example 3. If $\alpha \neq \beta$, but $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$ then find the quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.**

**Solution :** Given that $\alpha^2 = 5\alpha - 3$ and $\beta^2 = 5\beta - 3$

$\Rightarrow x^2 = 5x - 3$ or $x^2 - 5x + 3 = 0$ have the roots $\alpha$ and $\beta$

$\therefore \alpha + \beta = 5$ and $\alpha\beta = 3$

Now $$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{(5)^2 - 2(3)}{3} = \frac{19}{3}$$

and $$\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha} = 1$$

Hence, the required equation is :

$x^2$ – (sum of roots) $x$ + (product of roots) = 0

or $$x^2 - \frac{19}{3}x + 1 = 0$$

or $$3x^2 - 19x + 3 = 0$$

**Example 4. If $\alpha$ be one of the root of the equation $4x^2 + 2x - 1 = 0$ then prove that other root is $4\alpha^3 - 3\alpha$.**

**Solution :** Let $\beta$ be the other root of the equation

$$4x^2 + 2x - 1 = 0$$

then $$\alpha + \beta = -\frac{2}{4} \Rightarrow \beta = -\frac{1}{2} - \alpha \quad ....(1)$$

$\because$ $\alpha$ is the root of the equation $4x^2 + 2x - 1 = 0$

$\therefore$ $$4\alpha^2 + 2\alpha - 1 = 0 \Rightarrow \alpha^2 = \frac{1-2\alpha}{4} \quad ....(2)$$

Now
$$\begin{aligned} 4\alpha^3 - 3\alpha &= \alpha.4\alpha^2 - 3\alpha \\ &= \alpha(1-2\alpha) - 3\alpha \quad \text{[From (2)]} \\ &= -2\alpha^2 - 2\alpha \\ &= -2\left(\frac{1-2\alpha}{4}\right) - 2\alpha \\ &= -\alpha - \frac{1}{2} = \beta \quad \text{[From (1)]} \end{aligned}$$

Hence, the other root of the quadratic equation $4x^2 + 2x - 1 = 0$ is $\beta = 4\alpha^3 - 3\alpha$.

**Example 5. If $\alpha$ and $\beta$ are the roots of the equation $x^2 - p(x + 1) - c = 0$ then prove that $(\alpha + 1)(\beta + 1) = 1 - c$. Further prove that**

$$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = 1$$

**Solution :** $\therefore$ $\alpha$ and $\beta$ are roots of the equation

$$x^2 - px - (p + c) = 0$$

$\therefore$ $$\alpha + \beta = p \text{ and } \alpha\beta = -(p + c)$$

Now $(\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1 = -p - c + p + 1 = 1 - c$ ....(1)

Again $$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c} + \frac{\beta^2+2\beta+1}{\beta^2+2\beta+c} = \frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)} + \frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)} + \frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)(\beta+1)} \quad \text{[from (1)]}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)(\alpha-\beta)} + \frac{(\beta+1)^2}{(\beta+1)(\beta-\alpha)}$$

$$= \frac{\alpha+1}{\alpha-\beta} + \frac{\beta+1}{\beta-\alpha} = \frac{(\alpha+1)-(\beta+1)}{\alpha-\beta} = 1$$



**Example 6. If one root of a quadratic equation is $3 + \sqrt{7}$ then find the quadratic equation.** **[R.U. 2016]**

**Solution :**
$$\text{First root} = 3 + \sqrt{7}$$

$\therefore$ $$\text{other root} = 3 - \sqrt{7}$$

$$\text{Sum} = 6, \text{ product} = (3+\sqrt{7})(3-\sqrt{7}) = 9 - 7 = 2$$

$$\text{Equation } x^2 - 6x + 2 = 0$$

**Example 7. Arithmetic mean and geometric mean of two numbers are 5 and 4 respectively. Find these two numbers.** **[R.U. 2016]**

**Solution :** Let-two numbers be $\alpha$ and $\beta$

According to question $\frac{\alpha+\beta}{2} = 5$, $\sqrt{\alpha\beta} = 4$

$\therefore$ $\alpha + \beta = 10$ .... (1) and $\alpha\beta = 16$ ....(2)

$\Rightarrow$ $\beta = 10 - \alpha$

$\therefore$ From eq. (2) : $\alpha(10 - \alpha) = 16$

$\Rightarrow$ $10\alpha - \alpha^2 = 16$

$\Rightarrow$ $\alpha^2 - 10\alpha + 16 = 0$

$\Rightarrow$ $(\alpha - 2)(\alpha - 8) = 0$

$\Rightarrow$ $\alpha = 2$ or $8$

$\Rightarrow$ $\beta = 8$ or $2$

$\therefore$

$\therefore$ Two numbers are 2 and 8

**Example 8. A certain number of pens were purchased for Rs. 450. 5 more pens could have been purchased by the same amount if each per was cheaper by Rs. 15. Find the number of pens purchased.** **[R.U. 2016]**

**Solution :** Let number of pens $= x$

$\therefore$ $$\text{Cost of each pen} = \frac{450}{x}$$

If pens were cheaper by Rs. 15 then new cost of each pen

$$= \frac{450}{x} - 15$$

$$\text{Now number of pens} = x + 5$$

$\therefore$ $$(x+5)\left(\frac{450}{x} - 15\right) = 450$$

$\Rightarrow$ $(x + 5)(450 - 15x) = 450x$

$\Rightarrow$ $450x + 2250 - 15x^2 - 75x = 450x$

$\Rightarrow$ $15x^2 + 75x - 2250 = 0$

$$\Rightarrow \quad x^2 + 5x - 150 = 0$$
$$\Rightarrow \quad (x + 15)(x - 10) = 0$$
$$\Rightarrow \quad x = -15, 10$$
$$\Rightarrow \quad \text{Number of pens} = 10$$

**Example 9. If p and q are roots of a quadratic equation $2x^2 - 3x - 5 = 0$ then form a quadratic equation whose roots are $p^2$ and $q^2$. [R.U. 2016]**

**Solution :** Given equation $2x^2 - 3x - 5 = 0$

Roots are p and q

$$\therefore \quad p + q = \frac{3}{2}, \quad pq = \frac{-5}{2}$$

Now
$$p^2 + q^2 = (p + q)^2 - 2pq$$
$$= \frac{9}{4} - 2\left(-\frac{5}{2}\right)$$
$$= \frac{9}{4} + 5 = \frac{29}{4}$$

and
$$p^2q^2 = (pq)^2$$
$$= \left(-\frac{5}{2}\right)^2 = \frac{25}{4}$$

$\therefore$ Equation with roots $p^2$ and $q^2$ is

$$x^2 - \frac{29}{4}x + \frac{25}{4} = 0$$
$$\Rightarrow \quad 4x^2 - 29x + 25 = 0$$

## EXERCISES 7.5

1. Find the quadratic equation whose roots are :

   (i) $-2, -5$ (ii) $\frac{1}{2}, \frac{3}{2}$ (iii) 2, 3

2. Find the quadratic equation whose one root is $\frac{1}{3-\sqrt{2}}$

3. If $\alpha$ and $\beta$ are roots of the equation $ax^2 + bx + c = 0$, then find the values of following expressions :

   (i) $\alpha\beta^4 + \alpha^4\beta$ (ii) $(a\alpha + \beta)^{-2} + (a\beta + b)^{-2}$ (iii) $\alpha^{-2} + \beta^{-2}$

4. If $\alpha$ and $\beta$ are roots of the equation $x^2 - px + q = 0$, then prove that $\frac{\alpha^2}{\beta^2} + \frac{\beta^2}{\alpha^2} = \frac{p^4}{q^2} - \frac{4p^2}{q} + 2$.



5. If $\alpha$ and $\beta$ are the roots of the equation $x^2 - 2x + 3 = 0$ then find the equation whose roots are $\frac{\alpha - 1}{\alpha + 1}$ and $\frac{\beta - 1}{\beta + 1}$.

6. If the difference of the roots of the quadratic equation $x^2 + ax + b = 0$ is 1, then prove that $a^2 = 4b + 1$.

7. If $\alpha$ and $\beta$ are the roots of the equation $x^2 - px + q = 0$ then find the quadratic equation whose roots are $\frac{q}{p - \alpha}$ and $\frac{q}{p - \beta}$.

8. If the roots of the equation $ax^2 + bx + c = 0$ are sin $\alpha$ and cos $\alpha$, then prove that $a^2 - b^2 + 2ac = 0$.

## ANSWERS 7.5

1. (i) $x^2 + 7x + 10 = 0$ (ii) $4x^2 - 8x + 3 = 0$ (iii) $x^2 - 5x + 6 = 0$
2. $7x^2 - 6x + 1 = 0$
3. (i) $\frac{bc(3ac - b^2)}{a^4}$ (ii) $\frac{b^2 - 2ac}{c^2a^2}$ (iii) $\frac{b^2 - 2ac}{c^2}$
5. $3x^2 - 2x + 1 = 0$ 7. $x^2 - px + q = 0$

CbH