Chapter 3 Matrices

3.1 Introduction

In modern Mathematics matrix theory is used in various branches of pure and applied Mathematics. The theory of matrices has a special relationship with system of linear equations which occurs in many engineering processes.

Matrix-Definition

A set of mn numbers (real or complex) arranged in a rectangular array of m horizontal lines (rows) and n-vertical lines (columns) is known as Matrix of orer $m \times n$. These numbers are called elements, being enclosed in brackets [ ] or ‖ ‖. in a compact form the matrix is represented by $A = [a_{ij}]_{m \times n}$ where i = 1, 2, 3 ..... m and j = 1, 2, 3, ...n

An $m \times n$ matrix is usually written as

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{31} & a_{32} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Example :

$$A = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 5 & 9 \\ 3 & 1 & 6 \end{bmatrix}$$

then $a_{11} = 3$, $a_{12} = 2$, $a_{13} = 4$, $a_{21} = 2$, $a_{22} = 5$, $a_{23} = 9$, $a_{31} = 3$, $a_{32} = 1$, $a_{33} = 6$,

3.2 Types of Matrices

1. **Row Matrix :** A matrix having only one row is called row matrix.

   Example : $A = [2 \quad 1 \quad 3]_{1 \times 3}$

2. Column Matrix : A matrix having only one column is called column matrix.

Example : $A = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}_{3\times1}$

3. Square matrix : An $m \times n$ matrix for which $m = n$ is called a square matrix of order n i.e. equal number of rows and columns.

Example : $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & 5 & 6 \\ 8 & 6 & 7 \end{bmatrix}_{3\times3}$

4. Diagonal matrix : A squre matrix in which all the non diagonal elements are zero is called a diagonal matrix.

Example : $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{3\times3}$ or $A = \begin{cases} a_{ij} \neq 0 & \text{for } i = j \\ a_{ij} = 0 & \text{for } i \neq j \end{cases}$

5. Null Matrix : If every element of a matrix is zero then it is called null matrix i.e. $a_{ij} = 0, \forall\, i, j$

Example : $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$

6. Scalar Matrix : A square matrix in which all diagonal elements are equal and all other elements are zero is called scalar matrix. **[R.U. 2016]**

Example : $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times3}$

7. Identity Matrix : A square matrix in which each diagonal element is equal to unity and all other elements are zero is called identity matrix or unit matrix.

i.e. $A = \begin{cases} 0 & \text{when } i \neq j \\ 1 & \text{when } i = j \end{cases}$

Example : $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

Identity matrix of order $n \times n$ is denoted by I_n.

8. Upper Triangular Matrix : A square matrix in which all the elements below the leading digonal are zero is called an upper triangular matrix.

i.e. $a_{ij} = 0$ for $i > j$

Example : $A = \begin{bmatrix} 1 & 2 & 6 \\ 0 & 3 & 4 \\ 0 & 0 & 7 \end{bmatrix}_{3\times3}$

9. Lower triangular Matrix : A square matrix in which all the elements above the leading diagonal are zero is called a lower triangular matrix *i.e.* $a_{ij} = 0 \ \forall\ i < j$.

Example : $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 3 & 0 \\ 8 & 5 & 6 \end{bmatrix}_{3\times3}$

10. Idempotent Matrix : A square matrix A, such that $A^2 = A$ is called an idempotent matrix.

11. Involutory Matrix : A square matrix A such that $A^2 = I$ is called an involutory matrix.

3.3 Operations of Matrices

In this section, we will study various kinds of operations performed on matrices such as addition, subtraction, multiplication etc.

(I) Addition of Matrices : If A, B be two matrices, each of order m × n then their sum A + B is a matrix of order $m \times n$ and is obtained by adding the corresponding elements of A and B.

Thus, If $A = [a_{ij}]_{m\times n}$ and B $[b_{ij}]_{m\times n}$

then $C = A + B = [a_{ij} + b_{ij}]_{m\times n}$

for $i = 1, 2, \ldots m$ and $j = 1, 2, \ldots n$

$= [c_{ij}]_{m\times n}$

Example : If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 6 & 5 & 4 \\ 3 & 2 & 2 \end{bmatrix}$, then

$$C = A + B = \begin{bmatrix} 1+6 & 2+5 & 3+4 \\ 4+3 & 5+2 & 6+2 \end{bmatrix} = \begin{bmatrix} 7 & 7 & 7 \\ 7 & 7 & 8 \end{bmatrix}$$

The sum of two matrices is defined only when they are of the same order.

Example : If $A = \begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3}$

Then C = A + B is not defined, because A and B are not of the same order.

Properties of matrix addition

(i) Matrix-addition is commutative

$$A + B = B + A$$

(ii) Matrix addition is associative :

i.e. if A, B, C are three matrices of the same order,

then $(A + B) + C = A + (B + C)$

(II) Subtraction of Matrices : If A and B are two matrices of same order then (A – B) is obtained by subtracting each element of B from the corresponding elements of A.

i.e. For two matrices A and B of the same order, we define A – B = A + (– B)

Example : If $A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 7 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 5 & -2 \\ -1 & -4 & -7 \end{bmatrix}$

$$A - B = A + (-B) = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 4 & 7 \end{bmatrix} + \begin{bmatrix} 3 & -5 & 2 \\ 1 & 4 & 7 \end{bmatrix} = \begin{bmatrix} 6 & -3 & 3 \\ 2 & 8 & 14 \end{bmatrix}$$

(III) Matrix Multiplication : If A and B are two matrices then it is multiplible only when number of columns in A is equal to the number of rows in B.

Thus, if $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ are two matrices of order $m \times n$ and $n \times p$ respectively. then their product AB is of order $m \times p$ and is defined as

$$A \times B = (AB)_{ij} = \sum_{r=1}^{n} a_{ir}b_{rj} = a_{i1}b_{1j} + a_{i2}b_{2j} + ... + a_{in}b_{nj}$$

$$= [a_{i1}\ a_{i2}\ a_{in}] \times \begin{bmatrix} b_{2i} \\ b_{2i} \\ \vdots \\ b_{nj} \end{bmatrix} = (i^{th} \text{ row of A}) \times (j^{th} \text{ column of B})$$

$i = 1, 2, ...m$ and $j = 1, 2, ...p$

Example 1 : Find the product of $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} 5 & 6 \\ 2 & 0 \\ 5 & 1 \end{bmatrix}_{3\times2}$

Solution : $AB = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 3 & 1 \end{bmatrix} . \begin{bmatrix} 5 & 6 \\ 2 & 0 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 5.2+2.1+3.5 & 6.2+1.0+3.1 \\ 5.1+0.2+2.5 & 1.6+0.0+2.1 \\ 4.5+3.2+1.5 & 4.6+3.0+1.1 \end{bmatrix}_{3\times2}$

$$= \begin{bmatrix} 27 & 15 \\ 15 & 8 \\ 31 & 25 \end{bmatrix}_{3\times2} = C$$

Example 2 : If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ a & c & d \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$

Then find A × B, if possible.

Solution : Given $A = \begin{bmatrix} a & h & g \\ h & b & f \\ a & c & d \end{bmatrix}_{3\times3}$ and $B = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{3\times1}$

then A × B is possible since number of columns of A = number of rows of B.

Now $A \times B = \begin{bmatrix} a & h & g \\ h & b & f \\ a & c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ ax+cy+dz \end{bmatrix}_{3\times1}$

3.3.1 Non-commutativity of multiplication of matrices

Let A and B be two matrices such ath AB exists then it is quite possible that BA may not exist.

For example, if A is a 3 × 3 matrix and B is a 3 × 1 matrix then AB exist but BA does not exist. Similarly, if BA exists, then AB may not exist. Further, if AB and BA both exist, then they may not be equal.

Hence in general, AB ≠ BA

Example 3 : If $A = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$, find AB and BA and show that AB ≠ BA.

Here A is a 2 × 3 matrix and B is a 3 × 2 matrix, so AB exists and it is of order 2 × 2.

Now $A \times B = AB = \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix}$

$$= \begin{bmatrix} 2+2+12 & 3-4-15 \\ 6-2-4 & 9+4+5 \end{bmatrix} = \begin{bmatrix} 16 & -16 \\ 0 & 18 \end{bmatrix}_{2\times2}$$

Next, B is a 3 × 2 matrix and A is a 2 × 3 matrix, So, BA exists and it is of order 3 × 3.

Now $BA = \begin{bmatrix} 2 & 3 \\ -1 & 2 \\ 4 & -5 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 2+9 & -4+6 & 6-3 \\ -1+6 & 2+4 & -3-2 \\ 4-15 & -8-10 & 12+5 \end{bmatrix} = \begin{bmatrix} 11 & 2 & 3 \\ 5 & 6 & -5 \\ -11 & -18 & 17 \end{bmatrix}$$

Hence, AB ≠ BA

3.3.2 Existance of non-zero matrices whose product is the zero matrix (Restrict to square matrices or order)

If A and B are non-zero matrices but AB = 0

Example : $A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

while neither A nor B is the null matrix.

and $BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \neq 0$

i.e., in the case of matrix multiplication of AB = 0 then it does not necessarily imply that BA = 0

3.3.3 Multiplication of a Matrix by a Scalar

The multiplication of a matrix A by a scalar k is the matrix of same order as A and obtained by the multiplication of every element of matrix with k.

If matrix $A = [a_{ij}]_{m \times n}$ and k is any scalar

Then $kA = [ka_{ij}]_{m \times n}$

Example : The matrix $A = \begin{bmatrix} 3 & 4 & 1 \\ 7 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}_{3\times3}$ and k = 3

then $kA = 3\begin{bmatrix} 3 & 4 & 1 \\ 7 & 5 & 3 \\ 0 & 2 & 1 \end{bmatrix}_{3\times3} = \begin{bmatrix} 9 & 12 & 3 \\ 21 & 15 & 9 \\ 0 & 6 & 3 \end{bmatrix}_{3\times3}$

Properties

(i) If A and B are two matrices of same orders then, we have

$$k(A + B) = kA + kB, \text{ where k is any scalar.}$$

(ii) If k_1 and k_2 are two scalars and A is any matrix, then

$$(k_1+k_2)A = k_1A + k_2A$$

and $k_1(k_2A) = k_2(k_1A) = k_1k_2A = kA$ where $k = k_1k_2$

Example 4 : If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $A^2 - 4A - nI_2 = 0$, then find value of n.

Solution : Here $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

Then $A^2 = A.A \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} (2\times2)+(+1\times-1) & (2\times-1)+(-1\times2) \\ (-1\times2)+(2\times-1) & (-1\times-1)+(2\times2) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

Now $A^2 - 4A - nI_2 = 0$

i.e. $\begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} - 4\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - n\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 5-8-n & -4+4+0 \\ -4+4+0 & 5-8-n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -3-n & 0 \\ 0 & -3-n \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow -3-n=0 \Rightarrow n=-3$

Hence value of n = – 3.

Example 5 : If $A = \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}$, $X = \begin{bmatrix} n \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$ and AX = B, then n = ?

Solution : Given AX = B

$\Rightarrow \begin{bmatrix} 2 & 4 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} 2n+4 \\ 4n+3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \end{bmatrix}$

$\Rightarrow \left.\begin{matrix} 2n+4=8 \\ 4n+3=11 \end{matrix}\right\} \Rightarrow n=2$

Hence value of n = 2.

Example 6 : If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, then find value of A^2.

Solution : Given $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

Then $A^2 = A.A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$$= \begin{bmatrix} (1\times1)+(0\times0)+(0\times a) & (1\times1)+(0\times1)+(0\times b) & (1\times0)+(0\times0)+(0\times-1) \\ (0\times1)+(1\times0)+(0\times a) & (0\times0)+(1\times1)+(0\times b) & (0\times0)+(1\times0)+(0\times-1) \\ (a\times1)+(b\times0)+(-1\times a) & (a\times0)+(b\times1)+(-1\times b) & (a\times0)+(b\times0)+(-1\times-1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3 \text{ (Identity matrix of order 3).}$$

3.4 TRANSPOSE OF A MATRIX

Let $A = [a_{ij}]$ be an $m \times n$ matrix. Then the transpose of A, denoted by A^T or A', is an $n \times m$ matrix such that

$(A^T)_{ij} = [a_{ji}]$ for all $i = 1, 2, ...m; j = 1, 2, 3, ...n$

i.e. Transpose of a matrix is obtained by interchanging its rows and columns.

Example : If $A = \begin{bmatrix} 5 & 6 & 4 & 2 \\ 7 & 1 & 3 & 1 \\ 5 & 7 & 0 & 2 \end{bmatrix}_{3\times4}$ is a matrix of order 3 × 4.

Then, Its transpose is

$$A^T = \begin{bmatrix} 5 & 6 & 4 & 2 \\ 7 & 1 & 3 & 1 \\ 5 & 7 & 0 & 2 \end{bmatrix}^T_{3\times4} = \begin{bmatrix} 5 & 7 & 5 \\ 6 & 1 & 7 \\ 4 & 3 & 0 \\ 2 & 1 & 2 \end{bmatrix}_{4\times3}$$

note down

Properties of Transpose of a Matrix

If A and B are two matrices, then

(i) Transpose of the transpsoe of a matrix is the matrix itself.

i.e. $(A^T)^T = A$

(ii) $(A+B)^T = A^T + B^T$

(iii) $(kA)^T = kA^T$

(iv) $(AB)^T = B^TA^T$

Imp (v) If $AA^T = I = A^TA$ then matrix A is an Orthogonal matrix.

(vi) If $A = A^T$ then, square matrix A is called symmetric.

(vii) $A = -A^T$ then, square matrix A is skew-symmetric.

3.5 TRACE OF A MATRIX

Trace of a square matrix A is defined the sum of all diagonal elements of the matrix.

Example 7: If $A = \begin{bmatrix} 3 & 2 & 4 \\ 10 & 1 & 7 \\ 2 & 9 & 4 \end{bmatrix}_{3\times3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3}$

Then trace of $A = a_{11} + a_{22} + a_{33}$

$= 3 + 1 + 4 = 8$

question wrong

Example 8 : If $A = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix}$

verify that $(A + B)^T = A^T + B^T$

Solution : We have $A = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 3 & -2 & 7 \\ 4 & 0 & -5 \end{bmatrix}$

$B = \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix} \Rightarrow B^T = \begin{bmatrix} 2 & 5 & -1 \\ -3 & 6 & 8 \end{bmatrix}$

Now $A^T + B^T = \begin{bmatrix} 3 & -2 & 7 \\ 4 & 0 & -5 \end{bmatrix} + \begin{bmatrix} 2 & 5 & -1 \\ -3 & 6 & 8 \end{bmatrix} = \begin{bmatrix} 3+2 & -2+5 & 7+(-1) \\ 4-3 & 0+6 & -5+8 \end{bmatrix}$

$= \begin{bmatrix} 5 & 3 & 6 \\ 1 & 6 & 3 \end{bmatrix}$ .....(1)

$A + B = \begin{bmatrix} 3 & 4 \\ -2 & 0 \\ 7 & -5 \end{bmatrix} + \begin{bmatrix} 2 & -3 \\ 5 & 6 \\ -1 & 8 \end{bmatrix} = \begin{bmatrix} 3+2 & 4-3 \\ -2+5 & 0+6 \\ 7-1 & -5+8 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 3 & 6 \\ 6 & 3 \end{bmatrix}$

$\Rightarrow (A+B)^T = \begin{bmatrix} 5 & 1 \\ 3 & 6 \\ 6 & 3 \end{bmatrix}^T = \begin{bmatrix} 5 & 3 & 6 \\ 1 & 6 & 3 \end{bmatrix}$ .....(1)

Hence from (1) and (2) we get

$(A + B)^T = A^T + B^T$

Hence proved

3.6 Determinant of a Matrix

If $A = A = [a_{ij}]_{n\times n}$ is a square matrix of order n. Then the number $|a_{ij}|$ is called determinant of matrix A.

i.e. Every square matrix can be associated to an expression or a number which is known as its determinant. If $A = [a_{ij}]$ is a square matrix of order n, then the determinant of A is denoted by det A or |A|.

or $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times3} \Rightarrow |A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Expansion of a Determinant :

$|A| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{31} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$

Note :

(i) Only square matrices have determinants.

(ii) If a row or column of a determinant consists of all zeros, then the value of the determinant is zero.

(iii) The determinant of a square matrix can be expanded along any row or column.

3.6.1 Singular and Non-singular matrix

A squre matrix is a singular matrix if its determinant is zero otherwise, it is a non-singular matrix.

i.e. square matrix A is singular $\Rightarrow |A| = 0$

and square matrix A is non-singular $\Rightarrow |A| \neq 0$

Example 9: Evaluate (i) $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix}$

Solution : By definition

(i) $\begin{vmatrix} 5 & 4 \\ -2 & 3 \end{vmatrix} = (5 \times 3) - (4 \times -2) = 15 + 8 = 23$

(ii) $\begin{vmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{vmatrix} = (\sin\theta \times \sin\theta) - (\cos\theta \times \cos\theta)$

$= \sin^2\theta + \cos^2\theta = 1$

Example 10 : Find the determinant of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{bmatrix}$

Solution : $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 2 & 2 & 2 \end{vmatrix}$

$= 1\begin{vmatrix} 2 & 4 \\ 2 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix}$

$= 1(2 \times 2 - 4 \times 2) - 1(1 - 2 - 4 \times 2) + 1(1 \times 2 - 2 \times 2)$

$= 1(4 - 8) - 1(2 - 8) + 1(2 - 4)$

$= -4 + 6 - 2$

$= 0$

Example 11 : If $A = \begin{bmatrix} 3 & -2 & 4 \\ 1 & 2 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

$= (-1)^{1+1} . 3\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix} + (-1)^{1+2}(-2)\begin{vmatrix} 1 & 1 \\ 0 & -1 \end{vmatrix} + (-1)^{1+3} . 4\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$

$= 3(-2 - 1) + 2(-1 - 0) + 4(1 - 0)$

$= -9 - 2 + 4$

$= -7$

Ans.

Example 12 : Evaluate $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ by expanding it along the second row

Solution : By definition

$\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = (-1)^{2+1} .1\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + (-1)^{2+2} 2.\begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} + (1)^{2+3} .3\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$

minor no change

$= \begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + 2\begin{vmatrix} 2 & -2 \\ -2 & -3 \end{vmatrix} - 3\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix}$

$= -(-9 + 2) + 2(-6 - 4) - 3(2 + 6)$

$= 7 - 20 - 24 = -37$

Ans.

Example 13: Evalute the determinant $\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix}$ by expanding it along first column.

Solution : By definition

$\begin{vmatrix} 2 & 3 & -2 \\ 1 & 2 & 3 \\ -2 & 1 & -3 \end{vmatrix} = (-1)^{1+1} .2\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} + (-1)^{2+1} 1.\begin{vmatrix} 3 & -2 \\ 1 & -3 \end{vmatrix} + (-1)^{3+1}(-2)\begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix}$

$= 2(-6 - 3) - 1(-9 + 2) - 2(9 + 4)$

$= -18 + 7 - 26 = -37$

Ans.

3.7 Minors of a Matrix

Let $A = [a_{ij}]$ be a square matrix of order n. Then the minor M_{ij} of a_{ij} in A is the determinant of the square sub-matrix of order (n – 1) obtained by eleminating i^{th} row and j^{th} column of A.

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then the minor of $a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = M_{11}$

minor of $a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = M_{12}$

minor of $a_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = M_{13}$

minor of $a_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = M_{21}$

minor of $a_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = M_{22}$

minor of $a_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = M_{23}$

minor of $a_{31} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = M_{31}$

minor of $a_{32} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = M_{32}$

minor of $a_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = M_{33}$

Example 14 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{bmatrix}$

Then, $M_{11} = \text{minor of } (a_{11} = 1) = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix}; M_{12} = \text{Minor of } (a_{12} = 2) = \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix}$

$M_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix}, M_{21} = \begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix}, M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix}, M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}$

$M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix}, M_{32} = \begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix}, M_{33} = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix}$

3.8 Co-factors of any Element of a Square Matrix

The Co-factor of any element of the determinant is equal to the corresponding minor with a proper sign attached so it co-factor of element a_{ij} is denoted by C_{ij}.

i.e. $C_{ij} = (-1)^{i+j} M_{ij}$ where M_{ij} is minor of a_{ij} in A.

or Let $A = [A_{ij}]$ be a square matrix of order *n*. Then the co-factor C_{ij} of a_{ij} in matrix A is equal to $(-1)^{i+j}$ times the determinant of the sub-matrix of order (n −1) obtained by leaving i^{th} row and j^{th} column of A.

Thus, $C_{ij} = \begin{cases} M_{ij}, & \text{if } i+j \text{ is even} \\ -M_{ij}, & \text{if } i+j \text{ is odd} \end{cases}$

minor के चिन्ह को change करते हैं तो co-factor calculate होता है।

Example 15 : If $A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{bmatrix}$, then co-factors of element of A.

$C_{11} = (-1)^{1+1} M_{11} = M_{11} = \begin{bmatrix} 2 & -1 \\ -4 & 3 \end{bmatrix} = 2.$

$C_{12} = (-1)^{1+2} M_{12} = -\begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -[(-9+2)] = 7$

$C_{13} = (-1)^{1+3} M_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = (12 - 4) = 8$

$C_{21} = (-1)^{2+1} M_{21} = \begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix} = -(6 + 12) = -18$

$C_{22} = (-1)^{2+2} M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = (3 + 6) = -3$

$C_{23} = (-1)^{2+3} M_{23} = -\begin{vmatrix} 1 & 3 \\ 2 & -4 \end{vmatrix} = -(-4-4) = 8$

$C_{31} = (-1)^{3+1} M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = (-2 - 6) = -8$

$C_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} = -(-1 + 9) = -8$

$C_{33} = (-1)^{3+3} M_{33} = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = (2 + 6) = 8$

Note : Minors and co-factors are defined for the element of a square matrix.

3.9 Properties of Determinants

We mention some properties for a determinant of a square matrix $A = [a_{ij}]$ of order n.

(i) The sum of the product of elements of any row (column) with their co-factors is always equal so det (A) i.e.

$$\sum_{j=1}^{n} a_{ij}C_{ij} = |A| \text{ and } \sum_{i=1}^{n} a_{ij}C_{ij} = |A|$$

(ii) Then the sum of the product of elements of any row (column) with the cofactos of the corresponding elements of some other row (column) is zero i.e.

$$\sum_{j=1}^{n} a_{ij}C_{kj} = 0 \text{ and } \sum_{i=1}^{n} a_{ij}C_{ik} = 0$$

(iii) $|A| = |A^T|$

i.e. the value of a determinant remains unchanged if its rows and columns are interchanged.

(iv) If any two rows or columns of a determinant are identical then its value is zero.

(v) If each element of a row (column) of a determinant is multiplied by a constant k, then the value of the new determinant is k times the value of the original determinant.

i.e. $|kA| = k^n |A|$; A is a square matrix of order n.

(vi) If each element of a row (column) of a determinant is zero then its value is zero.

(vii) If $A = [a_{ij}]$ is a diagonal matrix of order n (≥ 2), then

$|A| = a_{11} . a_{22} . a_{33} . a_{nn}$

(viii) If A and B are square matrices of the same order, then

$|AB| = |A| . |B|$

Example 16 : Find the minor and co-factor of each element of $\begin{vmatrix} 1 & -3 & -2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$

Solution : Minors of the elements are given as

$$M_{11} = \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = (-2-10) = -12 \; ; \; M_{12} = \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = (8-6) = 2$$

$$M_{13} = \begin{vmatrix} 4 & -1 \\ 3 & 5 \end{vmatrix} = (20+3) = 23; \; M_{21} = \begin{vmatrix} -3 & 2 \\ 5 & 2 \end{vmatrix} = (-6-10) = -16$$

$$M_{22} = \begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} = (2+6) = -4 \; ; \; M_{23} = \begin{vmatrix} 1 & -3 \\ 3 & 5 \end{vmatrix} = 5+9 = 14$$

$$M_{31} = \begin{vmatrix} -3 & 2 \\ -1 & 2 \end{vmatrix} = -6+2 = -4; \; M_{32} = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = (2-8) = -6$$

$$M_{33} = \begin{vmatrix} 1 & -3 \\ 4 & -1 \end{vmatrix} = (-1+12) = 11$$

The cofactors of the corresponding elements are

$$C_{11} = (-1)^{1+1} \begin{vmatrix} -1 & 2 \\ 5 & 2 \end{vmatrix} = (-2-10) = -12; \; C_{12} = (-1)^{1+2} \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = -(8-6) = -2$$

$C_{13} = (-1)^{1+3} M_{13} = 23$, $C_{21} = (-1)^{1+4} M_{21} = 16$, $C_{22} = -4$, $C_{23} = -14$

$C_{31} = -4$, $C_{32} = 6$, $C_{33} = 11$.

3.10 Application of Determinant Infinding the area of a Triangle

We know that the area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by the expression :

$$\Delta = \frac{1}{2} [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]$$

$$\Rightarrow \Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ (In determinant form)

Since area is always a positive quantity, therefore we always take the absolute value of the determinant for the area.

If A (x_1, y_1), B(x_2, y_2) and C(x_3, y_3) are collinear ⇔ Area of triangle ABC = 0

i.e. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example 17 : Find the area of the triangle with vertices A(5, 4), B(–2, 4) and C(2,–6).

Solution : The area of triangle ABC $= \frac{1}{2} \begin{vmatrix} 5 & 4 & 1 \\ -2 & 4 & 1 \\ 2 & -6 & 1 \end{vmatrix}$

$$= \frac{1}{2} [5 (4+6) - 4 (-2-2) + 1 (12-8)]$$

$= \frac{1}{2}(50+16+4) = \frac{1}{2}(70) = 35$ sq. units.

Example 18 : Show tha the points $(a, b+c)$, $(b, c+a)$ and $(c, a+b)$ ae collinear.

Solution : $(x_1, y_1) = (a, b+c)$, $(x_2, y_2) = (b, c+a)$ and $(x_3, y_3) = (c, a+b)$

then $$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}$$

Expanding for third column

$$= 1\begin{vmatrix} b & c+a \\ c & a+b \end{vmatrix} - 1\begin{vmatrix} a & b+c \\ c & a+b \end{vmatrix} + 1\begin{vmatrix} a & b+c \\ b & a+a \end{vmatrix}$$

$= [b(a+b) - c(c+a)] - [a(a+b) - c(b+c)] + [a(c+a) - b(b+c)]$

$= ab + b^2 - c^2 - ac - a^2 - ab + bc + c^2 + ac + a^2 - b^2 - bc$

$= 0$

Hence, the given points are collinear.

3.11 Adjoint of a Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in A. Then the transpose of the matrix of cofactors of elements of A is called the adjoint of A and is denoted by adj A.

If $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

where C_{ij} denotes the cofactor of a_{ij} in A.

Note : The adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing signs of off-diagonal elements.

Example 19: If $A = \begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$

* cofactor transpose ⇒ adjA

$A^{-1} = \frac{\text{adjA}}{|A|}$

then by above rule

$$\text{adj } A = \begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$$

Example 20 : Find the adjoint of matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 3 \end{bmatrix}$

Solution : Cofactors of elements of A are given by :

$C_{11} = \begin{vmatrix} 1 & -3 \\ 2 & 2 \end{vmatrix} = 9$; $C_{12} = \begin{vmatrix} 2 & -3 \\ -1 & 2 \end{vmatrix} = -3$, $C_{13} = \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 5$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -1$; $C_{22} = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$; $C_{23} = \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} = -3$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} = -4$; $C_{32} = -\begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = 5$; $C_{33} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$

$$\therefore \text{ adj } A = \begin{bmatrix} 9 & -3 & 5 \\ -1 & 4 & -3 \\ -4 & 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 9 & -1 & -4 \\ -3 & 4 & 5 \\ 5 & -3 & -1 \end{bmatrix}$$

Example 21 : Compute the adjoint of the matrix.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 6 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution : The cofactors of elements of A are given by

$C_{11} = \begin{vmatrix} 2 & 6 \\ 1 & 0 \end{vmatrix} = -6$, $C_{12} = -\begin{vmatrix} 3 & 6 \\ 0 & 0 \end{vmatrix} = 0$, $C_{13} = \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} = 3$

$C_{21} = -\begin{vmatrix} 4 & 5 \\ 1 & 0 \end{vmatrix} = +5$, $C_{22} = \begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix} = 0$, $C_{23} = -\begin{vmatrix} 1 & 4 \\ 0 & 1 \end{vmatrix} = -1$

$C_{31} = \begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} = 24 - 10 = 14$; $C_{32} = -\begin{vmatrix} 1 & 5 \\ 3 & 6 \end{vmatrix} = 9$; $C_{33} = \begin{vmatrix} 1 & 4 \\ 3 & 2 \end{vmatrix} = -10$

$$\text{Now adj } A = \begin{bmatrix} -6 & 0 & 3 \\ 5 & 0 & -1 \\ 14 & 9 & -10 \end{bmatrix}^T = \begin{bmatrix} -6 & 5 & 14 \\ 0 & 0 & 9 \\ 3 & -1 & -10 \end{bmatrix}$$

3.12 Invertible Matrices

A square matrix A of order n is invertible if there exists a square matrix B of the same order such that

$$AB = I_n = BA$$

Note : A square matrix is invertible iff it is non-singular.

3.13 Inverse of a Matrix

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The inverse of a non-singular square matrix A is denoted by A^{-1} and defined by

$$A^{-1} = \frac{adj\ A}{|A|}$$

The inverse of a matrix A will exist if $AA^{-1} = I$.

where I is the unit matrix.

If matrix is singular then its inverse does not exist since for singular matrix, we have $|A| = 0$.

Example 22 : Find the inverse of matrix by adjoint method

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution : Given $A = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{vmatrix} = 1(2-4)-2(6-4)+5(3-1)$$

$$= -2-4+10 = 4$$

Now, cofactors $C_{11} = -2, C_{12} = -2, C_{13} = 2, C_{21} = 1, C_{22} = -3, C_{23} = 1$
$C_{31} = 3, C_{32} = 11, C_{33} = -5.$

Therefore, the matrix of cofactors $C = \begin{bmatrix} -2 & -2 & 2 \\ 1 & -3 & 1 \\ 3 & 11 & -5 \end{bmatrix}$

$$\therefore \quad adj\ A = C^T = \begin{bmatrix} -2 & 1 & 2 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$$

Now $$A^{-1} = \frac{adj\ A}{|A|} = \frac{1}{4}\begin{bmatrix} -2 & 1 & 2 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 & 3/4 \\ -1/2 & -3/4 & 11/4 \\ 1/2 & 1/4 & -5/4 \end{bmatrix}$$

3.14 Solutions of a System of Linear Equations

In this section we apply the theory of matrices to study the nature of solutions and existence for a system of m linear equations in n unknowns

Consider the system of m linear equations in n unknowns.

$x_1, x_2, \ldots\ldots x_n$; given by

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n = b_m$$

These set of euations can be written in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m_1} & a_{m_2} & \cdots & a_{m_n} \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

i.e. $AX = B$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m_1} & a_{m_2} & \cdots & a_{m_n} \end{bmatrix}$; $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$; $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

Here A is coefficient matrix, X is the matrix of unknown and B i the column matrix of the constants.

Types of Linear equations

1. **Homogeneous linear equations :** If $b_1 = b_2 = \ldots\ldots = b_m = 0$ then $B = 0$ and the matrix equation $AX = B$ reduces to $AX = 0$, which is called homogeneous equations.

2. Non-homogeneous linear equations : If at least one of $b_1, b_2 \ldots\ldots b_m$ is non-zero then $B \neq 0$ and the matrix equation on AX = B is known is non-homogeneous equations.

Solution of Linear equations

A set of unknowns $x_1, x_2 \ldots x_n$ which satisfies all the equations of AX = B is called solution.

1. **Consistent** : A system of equations AX = B is said to be consistent if the system have a solution.

2. **Inconsistent** : A system of equations AX = B is said to be inconsistent if the system have no solution.

3.15 Solution of sytem of linear equations by inverse matrix method (matrix method) :

Let AX = B be a system of n linear equations with n uknowns. If A is non-singular ($|A| \neq 0$), then A^{-1} exiss.

Thus, the system of equations AX = B has a solution given by

$$X = A^{-1}.B$$

Algorithm for solving a non-homogeneous system of linear equations

Let AX = B be a non-homogeneous system of linear equations.

Step-I : Write the given system of equations in matrix form AX = B and obtain A, B.

Step-II : Find |A|

Step-III : If $|A| \neq 0$, then the system is consistent with unique solution, then find A^{-1} by using $A^{-1} = \dfrac{adj\,A}{|A|}$

obtain the unique solution given by $X = A^{-1}B$

Step-IV : If |A| = 0, then, the system is either consistent with infinitely many solutions or it is inconsistent find (adj A)B

If (adj A) $B \neq 0$, the system is inconsistent.

If (adj A)B = 0, then, the system is consistent with infinitely, many solutions.

For solutions put z = k and take any two equation out of three equations. Solve these equations for x and y.

Let the values of x and y be λ and μ respectively.

Then, x = λ, y = μ, z = k is the required solution.

Algorithm for solving a homogeneous system of linear equations

Step-I : Write the given system of equations in matrix form AX = 0 and obtain A.

Step-II : Find |A|

Step-III : If $|A| \neq 0$, then the system is consistent with unique solution x = y = z = 0 (complete).

Step-IV : If |A| = 0, the system of equations has infinitely many solutions. To find these solutions proceed as follows. Put z = k and solve any two equations for x and y in terms of k.

Example 23 : Solve the system of non-homogeneous equations $x + y + z = 8$, $2x + 3y + 2z = 19$, $4x + 2y + 3z = 23$, using inverse matrix method.

Solution : Given system of linear equations

$$x + y + z = 8$$
$$2x + 3y + 2z = 19$$
$$4x + 2y + 3z = 23$$

Step-I : Equation can be written in matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \\ 23 \end{bmatrix}$$

i.e. AX = B

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix}; B = \begin{bmatrix} 8 \\ 19 \\ 23 \end{bmatrix}$$

Step-II :

$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 4 & 2 & 3 \end{bmatrix} = 1(9-4) - 1(6-8) + 1(4-12)$$
$$= 5 + 2 - 8 = -1 \neq 0$$

Step-III : $\because$ $|A| \neq 0$ i.e. the system is consistent with unique solution. Now cofactors of elements of A are

$C_{11} = 5, C_{12} = 2, C_{13} = -8, C_{21} = -1, C_{22} = -1, C_{23} = 2$

$C_{31} = -1, C_{32} = 0, C_{33} = 1$

Hence
$$adj\,A = \begin{bmatrix} 5 & 2 & -8 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 5 & -1 & -1 \\ 2 & -1 & 0 \\ -8 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{adj\,A}{|A|} = \frac{1}{-1}\begin{bmatrix} 5 & -1 & -1 \\ 2 & -1 & 0 \\ -8 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 1 \\ -2 & 1 & 0 \\ 8 & -2 & -1 \end{bmatrix}$$

Solution : $X = A^{-1} B$

$$\Rightarrow \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 & 1 & 1 \\ -2 & 1 & 0 \\ 8 & -2 & -1 \end{bmatrix}\begin{bmatrix} 8 \\ 19 \\ 23 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$$

i.e. $x = 2, y = 3, z = 3$.

Example 24 : Find the solution of the following system of equations $5x + 3y + 7z = 4$, $3x + 26y + 2z = 9$, $7x + 2y + 10z = 5$

Solution : Given equation are

$$5x + 3y + 7z = 4$$
$$3x + 26y + 2z = 9$$
$$7x + 2y + 10z = 5$$

In matrix form
$$\begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix} = 5(260 - 4) - 3(30 - 14) + 7(6 - 182)$$
$$= 1280 - 48 - 1232$$
$$= 1280 - 1280 = 0$$

Now, cofactors of element of matrix.

$C_{11} = 260 - 4, 256, C_{12} = -16, C_{13} = -176, C_{21} = -16, C_{22} = 1$

$C_{21} = 11, C_{31} = -176, C_{32} = 11, C_{33} = 121$

$$adj\,A = \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix}^T = \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix}$$

$$(adj\,A)\,B = \begin{bmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{bmatrix}\begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ i.e. } (adj\,A)B = 0$$

i.e. the system is consistent with infinity many solutions.

Now, putting $z = k$, in given equations, we get

$$5x + 3y = 4 - 7k$$
$$7x + 2y = 5 - 10k$$

Solving, we get

$$x = \frac{7 - 16k}{11}, y = \frac{3 + k}{11}$$

Hence $x = \frac{7 - 16k}{11}$, $y = \frac{3 + k}{11}$, $z = k$ are the solution.

Example 25 : Solve the following system of homogeneous equations $2x + 3y - z = 0$, $x - y - 2z = 0$, $3x + y + 3z = 0$

Solution : The given system of equations

$$2x + 3y - z = 0$$
$$x - y - 2z = 0$$
$$3x + y + 3z = 0$$

In matrix form

$$\begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

where
$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Now
$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & -1 & -2 \\ 3 & 1 & 3 \end{vmatrix} = 2(-3 + 2) - 3(3 + 6) - 1(1 + 3)$$
$$= -2 - 27 - 4 = -33 \neq 0$$

The system is consistent with unique solution.

$$x = y = z = 0$$

Ans.

Example 26 : Solve the following system of homgeneuos equatins $x - 2y + z = 0$, $x + y - z = 0$, $3x + 6y - 5z = 0$

Solution : Given equations are

$$x - 2y + z = 0$$
$$x + y - z = 0$$
$$3x + 6y - 5z = 0$$

In matrix form

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$AX = 0$$

where $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$|A| = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \\ 3 & 6 & -5 \end{bmatrix} = 1(-5+6) + 2(-5+3) + 1(6-3)$$

$$= 1 - 4 + 3 = 0$$

i.e. $|A| = 0$

The system of equations has infinitely many solutions. Putting $z = k$ in given equations.

$$\left.\begin{aligned} x - 2y &= -k \\ x + y &= k \\ 3x + 6y &= 5k \end{aligned}\right\} \Rightarrow x = \frac{k}{3}, y = \frac{2k}{3}$$

Hence $x = \frac{k}{3}$, $y = \frac{2k}{3}$, $z = k$ are the solution.

3.16 Solution of system of linear equations by Cramer's rule

Cramer's Rule : Let there be a system of n simultaneous linear equation in n

unknowns as, given by

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \ldots\ldots + a_{nn}x_n = b_n$$

$$\text{Let } D = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n_1} & a_{n_2} & \cdots & a_{n_n} \end{bmatrix}; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

and let D_j be the determinant obtained from D after replacing the j^{th} column by B.

Then, $x_1 = \frac{D_1}{D}$, $x_2 = \frac{D_2}{D}$, ... $x_n = \frac{D_n}{D}$, provided $D \neq 0$

Algorithm for solving a system of simultaneous linear equations By Cramer's Rule (Determinant Method) :

Step-I : Obtain D, D_1, D_2 and D_3

Step-II : Find value of D, D_1, D_2 and D_3

If $D \neq 0$, the system of equations is consistent and has a unique solution, Then solution is given by :

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D}$$

Step-III : If $D = 0$, $D_1 = D_2 = D_3 = 0$

Then for solution, take any two equations out of three given equations and shift the variable z on the right hand side to obtain two equations in x, y. Solve these two equations by Cramer's rule so obtain x, y in terms of z.

If $D = 0$ and at least one of these determinants in non-zero, then the system is inconsistent.

Example 27 : Solve by Cramer's rule

$$2x - y = 17$$
$$3x + 5y = 6$$

Solution : $D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix}$, $D_1 = \begin{vmatrix} 17 & -1 \\ 6 & 5 \end{vmatrix}$, $D_2 = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix}$

$\Rightarrow$ D = 10 + 3 = 13; D_1 = 85 + 6 = 91 ; D_2 = 12 – 51 = – 39

So, by Cramer's rule, we have

$$x = \frac{D_1}{D} = \frac{91}{13} = 7 \text{ and } y = \frac{D_2}{D} = \frac{-39}{13} = -3.$$

Hence $x = 7$ and $y = -3$ is the required solution.

Example 28 : Solve the following system of equations using Cramer's Rule.

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

Solution : The given system of equations is

$$5x - 7y + z = 11$$
$$6x - 8y - z = 15$$
$$3x + 2y - 6z = 7$$

$$\therefore\ D = \begin{vmatrix} 5 & -7 & 1 \\ 6 & -8 & -1 \\ 3 & 2 & -6 \end{vmatrix}; D_1 = \begin{vmatrix} 11 & -7 & 1 \\ 15 & -8 & -1 \\ 7 & 2 & -6 \end{vmatrix}; D_2 = \begin{vmatrix} 5 & 11 & 1 \\ 6 & 15 & -1 \\ 3 & 7 & -6 \end{vmatrix};$$

$$D_3 = \begin{vmatrix} 5 & -7 & 11 \\ 6 & -8 & 15 \\ 3 & 2 & 7 \end{vmatrix}$$

Solving, we get

$\Rightarrow$ D = 55 ≠ 0, D_1 = 55, D_2 = – 55 and D_3 = – 55

so, By Cramer's Rule

$$x = \frac{D_1}{D} = \frac{55}{55} = 1;\ y = \frac{D_2}{D} = \frac{-55}{55} = -1;\ z = \frac{D_3}{D} = \frac{-55}{55} = -1;$$

Hence x = 1, y = – 1, and z = – 1, is the required solution of the given system of equations.

Example 29 : Solve the system of equations $x + 2y = 3$, $4x + 8y = 12$ by using determinants method.

Solution : The given system of equations is

$$x + 2y = 3$$
$$4x + 8y = 12$$

$$\therefore\ D = \begin{vmatrix} 1 & 2 \\ 4 & 8 \end{vmatrix}, D_1 = \begin{vmatrix} 3 & 2 \\ 12 & 8 \end{vmatrix}, D_2 = \begin{vmatrix} 1 & 3 \\ 4 & 12 \end{vmatrix}$$

$\Rightarrow$ D = 8 – 8 = 0; D_1 = 24 – 24 = 0; D_2 = 12 – 12 = 0

Thus, D = D_1 = D_2 = 0

So, the given system of equations has infinite number of solutions.

Let y = k, then $x = 3 - 2y - 3 - 2k$.

Hence, $x = 3 - 2k$, $y = k$ is the solution of the given system of equations, where k is an arbitrary real number.

Example 30: By using determinants, solve the following system of equations.
$x + y + z = 1$, $x + 2y + 3z = 4$, $x + 3y + 5z = 7$

Solution : The given system of equations

$$x + y + z = 1$$
$$x + 2y + 3z = 4$$
$$x + 3y + 5z = 7$$

$$\therefore\ D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{vmatrix}; D_1 = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 3 \\ 7 & 3 & 5 \end{vmatrix}; D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 4 & 3 \\ 1 & 7 & 5 \end{vmatrix}; D_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 7 \end{vmatrix}$$

$\Rightarrow$ D = 0, D_1 = 0, D_2 = 0, D_3 = 0

$\therefore$ the given system of equations has infinitely many solutions. From first two equations.

$$x + y = 1 - z$$
$$x + 2y = 4 - 3z$$

For solution, we use Cramer's rule

$$D = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = (2 - 1) = 1, D_1 = \begin{vmatrix} 1-z & 1 \\ 4-3z & 2 \end{vmatrix} = (2 - 2z) - (4 - 3z) = -2 + z$$

$$\text{and } D_2 = \begin{vmatrix} 1 & 1-z \\ 1 & 4-3z \end{vmatrix} (4 - 3z) - (1 - z) = 3 - 2z.$$

$$\therefore\ x = \frac{D_1}{D} = \frac{z-2}{1} = z - 2;\ y = \frac{D_2}{D} = \frac{3-2z}{1} = 3 - 2z$$

let $z = k$, where k is any real number.

Then , $x = k - 2$, $y = 3 - 2k$, $z = k$

Example 31 : Using determinants method, solve the following system of equations.

$2x - y + 3 = 4, x + 3y + 2z = 12, 3x \; 2y + 3z = 10$

Solution : The given system of equations is

$$2x - y + 3 = 4$$
$$x + 3y + 2z = 12$$
$$3x \; 2y + 3z = 10$$

$$\therefore \quad D = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 2 & 3 \end{vmatrix} = 2(9-4) + 1(3-6) + 1(2-9)$$

$$= 10 - 3 - 7 = 0$$

$$D_1 = \begin{vmatrix} 4 & -1 & 1 \\ 12 & 3 & 2 \\ 10 & 2 & 3 \end{vmatrix} = 4(9-4) 1 + (36-20) + 1(24-30)$$

$$= 20 + 16 - 6 = 30 \neq 0$$

Hence, the given system of equations is inconsistent.

3.17 Eigen values and Eigen vectors of a square matrix

Let A be a $n \times n$ matrix. Suppoes the linear transformation $y = AX$ transforms X into a scalar multiple of it self.

i.e. $AX = y = \lambda X$

$\Rightarrow \quad (A - \lambda I) X = 0$

Then the unknown scalar λ is known as an eigen value of the matrix A and the corresponding non-zero vector X is known as eigen vector of A.

This system of equations has non-trivial solutions if the coefficient matrix $(A - \lambda I)$ is singular i.e.

$$|A - \lambda I| = 0$$

This equation is known as characteristic equation of A.

Note :

(i) Eigen values of a square matrix A are roots of charactristic equation.

(ii) If all the eigen values of a matrix A are distinct then the corresponding eigen vectors are linearly independent.

(iii) If A is singular then at least one of its eigen value is zero.

Example 32: Find the eigen values and corresponding eigen vectors of the matrix.

$$\begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$

Solution : Given matrix is $A = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$

Let λ be the eigen value of A.

Then, the characteristic equation of A is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\Rightarrow (8-\lambda)\{(7-\lambda)(3-\lambda) - 16\} + 6\{(3-\lambda) \times (-6) + 8\}\{24 - 2(7-\lambda)\} = 0$

$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0$

or $\lambda(\lambda - 3)(\lambda - 15) = 0$

$\Rightarrow \lambda = 0, 3, 15$

$\therefore$ Eigen values of A are 0, 3, 15

To find Eigen Vectors :

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to λ.

Then $\quad AX = \lambda x$

$\Rightarrow \quad (A - \lambda I) X = 0$

or $$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad .....(1)$$

Case-I : Putting $\lambda = 0$ in equation (1), we get

$$8x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 7x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$

Solving, by rule or cross multiplication, we have

$$\frac{x_1}{24-14} = \frac{x_2}{32-12} = \frac{x_3}{56-36}$$

or $$\frac{x_1}{10} = \frac{x_2}{20} = \frac{x_3}{20}$$

or $$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{2} = k_1 \ (k_1 \neq 0)$$

$\Rightarrow$ $$x_1 = k_1, x_2 = 2k_1, x_3 = 2k_1$$

$\therefore$ Eigen vector corresponding to $\lambda = 0$ is $x_1 = \begin{bmatrix} k_1 \\ 2k_2 \\ 2k_1 \end{bmatrix}$

Case-II : Put $\lambda = 3$ in equation (1), we get

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$5x_1 - 6x_2 + 2x_3 = 0$$
$$-6x_1 + 4x_2 - 4x_3 = 0$$
$$2x_1 - 4x_2 + 0x_3 = 0$$

Solving, we get

$$\frac{x_1}{24-8} = \frac{x_2}{20-12} = \frac{x_3}{20-36}$$

$\Rightarrow$ $$\frac{x_1}{16} = \frac{x_2}{8} = \frac{x_3}{-16}$$

$\Rightarrow$ $$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k_2 \ (k_2 \neq 0)$$

$\Rightarrow$ $$x_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_1 \\ k_2 \\ -2k_2 \end{bmatrix}$$

Case-III : Putting $\lambda = 5$ in equation (1), we get

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving, we get

$$\frac{x_1}{24+16} = \frac{x_2}{-12-28} = \frac{x_3}{56-36}$$

$\Rightarrow$ $$\frac{x_1}{2} = \frac{x_2}{-2} = \frac{x_3}{1} = k_3 \ (k_3 \neq 0)$$

$\Rightarrow$ $$\frac{x_1}{2} = \frac{x_2}{1} = \frac{x_3}{-2} = k_2 \ (k_2 \neq 0)$$

$\Rightarrow$ $$x_3 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_3 \\ -2k_3 \\ k_3 \end{bmatrix}$$

$\therefore$ Eigen vectors corresponding to $\lambda = 15$ is $X_3 = \begin{bmatrix} 2k_3 \\ -2k_3 \\ k_3 \end{bmatrix}$

Example 33: If A is any $m \times n$ matrix such that AB and BA are both defined then find order of matrix B.

Solution : Since AB exists

Therefore number of rows in B = number of columns in A

$\therefore$ B has n rows

Now BA exists

$\Rightarrow$ number of columns in B = number of rows in A

$\Rightarrow$ B has m columns

Hence B is of order $n \times m$

Example 34: If $\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$, then find matrix A.

Solution : Given equation

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix} \quad \{\because AB = C \Rightarrow A = B^{-1}C\}$$

$$= \begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}$$

$$= \begin{bmatrix}(1\times1)+(-3\times0) & (1\times1)+(-3\times-1)\\(0\times1)+(1\times0) & (0\times1)+(1\times-1)\end{bmatrix} = \begin{bmatrix}1 & 4\\0 & -1\end{bmatrix}$$ **Ans.**

Example 33 : If A and B are two invertible matrices, then find inverse of matrix AB.

Solution : ∵ A and B are invertible

So A^{-1} and B^{-1} are both exists.

Now, $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$ $\{\because BB^{-1} = I\}$

$= AIA^{-1}$

$= (AA^{-1}) = I$ $\{\because AI = A\}$

and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$

$= B^{-1}IB$

$= B^{-1}B = I$

∴ $(AB)(B^{-1}A^{-1}) = I(B^{-1}A^{-1})(AB)$

⇒ AB is inversible

⇒ $(AB)^{-1} = B^{-1}A^{-1}$ **Ans.**

Example 36: If $A = \begin{bmatrix}1 & -5 & -7\\0 & 7 & 9\\11 & 8 & 9\end{bmatrix}$, then find trace of matrix A.

Solution : If $A = [a_{ij}]$ is a square matrix

then trace of $A = \sum_i a_{ii}$

so, for $A = \begin{bmatrix}1 & -5 & -7\\0 & 7 & 9\\11 & 8 & 9\end{bmatrix}$

$a_{11} = 1, a_{22} = 7, a_{33} = 9$

∴ Trace of $A = tr(A) = a_{11} + a_{22} + a_{33}$

$= 1 + 7 + 9 = 17$ **Ans.**

Example 37 : If A is a singular matrix, then find value of A (adj A)

Solution : A is singular ⇒ $|A| = 0$

we know that

$A(\text{adj } A) = A(A^{-1}|A|)$ $\left\{\because A^{-1} = \frac{adj A}{|A|}\right.$

$= |A|\, I$ $\{AA^{-1} = I\}$

$= I = 0$

Hence A(adj A) is the null matrix.

Example 38: If $A = \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}$ and $A^2 - kA - 5I_2 = 0$, then calculate the value of k.

Solution : Given $A = \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}$

$A^2 - kA - 5I_2 = 0$

⇒ $kA = A^2 - 5I_2$

$$= \begin{bmatrix}1 & 3\\3 & 4\end{bmatrix}\begin{bmatrix}1 & 3\\3 & 4\end{bmatrix} - 5\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}5 & 15\\15 & 20\end{bmatrix}$$

$$= 5\begin{bmatrix}1 & 3\\3 & 4\end{bmatrix} = 5A$$

⇒ $kA = 5A$

⇒ $k = 5.$

Example 39 : If $A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$; then find value of A^4.

Solution : $A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$$A^2 = A.A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$A^4 = A^2.A^2 = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I_2$$

Example 40 : The system of linear equations $x + y + z = 2$, $2x + y - z = 3$, $3x + 2y + kz = 4$. Find value of k for which system of equation has an unique solution.

Solution : Given system of equations

$$x + y + z = 2$$
$$2x + y - z = 3$$
$$3x + 2y + kz = 4$$

The given system of equations has a unique solution if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow k \neq 0$$

Example 41 : Simplify

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

Solution : We have

$$\cos\theta \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} + \sin\theta \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos^2\theta \end{bmatrix} + \begin{bmatrix} \sin^2\theta & -\cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \text{ (Identify matrix of order 2)}$$

Example 42 : Find the value of x such that

$$[1 \times 1] \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

Solution : We have

$$[1 \times 1] \begin{bmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

$$\Rightarrow [1 + 2x + 15 \quad 3 + 5x + 3 \quad 2 + x + 2] \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

$$\Rightarrow [2x + 16 \quad 5x + 6 \quad x + 4] \begin{bmatrix} 1 \\ 2 \\ x \end{bmatrix} = 0$$

$\Rightarrow [16 + 2x + 2(6 + 5x) + x(4 + x)] = 0$

$\Rightarrow x^2 + 16x + 28 = 0$

$\Rightarrow x^2 + 14x + 2x + 28 = 0$

$\Rightarrow x(x + 14) + 2(x + 14) = 0$

$\Rightarrow (x + 2)(x + 14) = 0$

Hence $x = -2, -14$

Example 43 : If $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$, verify that

$$A.(adiA) = (adjA).A = |A|I_3$$

Solution : We have $A = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{vmatrix}$$

$= 1(-28 + 30) - 0(-21 - 0) - 1(18 - 0)$

$= 2 + 0 + 18 = 20$

Now co-factor's of elements of A are

$C_{11} = 2, C_{12} = 21, C_{13} = -18, C_{21} = +6, C_{22} = -7, C_{23} = 6, C_{31} = 4, C_{32} = -8, C_{33} = 4$

$$adjA = \begin{bmatrix} 2 & 21 & -18 \\ 6 & -7 & 6 \\ 4 & -8 & 4 \end{bmatrix}^T = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

Now $$A(adjA) = \begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix} = 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 20I_3 = |A|I_3$$

$$(adjA)A = \begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{bmatrix}$$

$$= 20\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 20I_3 = |A|I$$

$\therefore$ $$A(adjA) = (adjA)A = |A|I_3$$

Example 44 : Show that the matrix $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $A^2 - 4A - I_3 = 0$, Hence find A^{-1}

Solution : Given $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Now, $$A^2 = A.A. = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \text{ Hence proved}$$

Further, $A^2 - 4A - 5I = 0$

$\Rightarrow$ $A - 4I - 5A^{-1} = 0$

$\Rightarrow$ $$A^{-1} = \frac{1}{5}[A - 4I]$$

$$= \frac{1}{5}\left\{\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$= \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Example 45 : Solve the following equations by cramer's rule.

$3x + y = 19$

$3x - y = 23$

Solution : Given equations are

$3x + y = 19$

$3x - y = 23$

$$D = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -3 - 3 = -6,\ D_1 = \begin{vmatrix} 19 & 1 \\ 23 & -1 \end{vmatrix} = -19 - 23 = -42$$

$$D_2 = \begin{vmatrix} 3 & 19 \\ 3 & 23 \end{vmatrix} = 69 - 57 = 12$$

Since $D \neq 0$; so system has unique, solution. By cramer's rule, we have

$$x = \frac{D_1}{D} = \frac{-42}{-6} = 7,\ y = \frac{D_2}{D} = \frac{12}{-6} = -2$$

Hence, solution of given equations $x = 7, y = -2$

Example 46.

$3x + y + z = 2$
$2x - 4y + 3z = -1$
$4x + y - 3z = -11$

Solve the given system of equation by cramer's rule. **[R.U. 2015]**

Sol. :

$$\Delta = \begin{vmatrix} 3 & 1 & 1 \\ 2 & -4 & 3 \\ 4 & 1 & -3 \end{vmatrix} = 63$$

$$\Delta_1 = \begin{vmatrix} 2 & 1 & 1 \\ -1 & -4 & 3 \\ -11 & 1 & -3 \end{vmatrix} = -63$$

$$\Delta_2 = \begin{vmatrix} 3 & 2 & 1 \\ 2 & -1 & 3 \\ 4 & -11 & -3 \end{vmatrix} = 126$$

$$\Delta_3 = \begin{vmatrix} 3 & 1 & 2 \\ 2 & -4 & -1 \\ 4 & 1 & -11 \end{vmatrix} = 63$$

$\therefore$ $$x = \frac{\Delta_1}{\Delta} = -\frac{63}{63} = -1$$

$$y = \frac{\Delta_2}{\Delta} = \frac{126}{63} = 2$$

$$z = \frac{\Delta_3}{\Delta} = \frac{63}{63} = 1$$

Example 47. If $\begin{vmatrix} 3 & -1 \\ 1 & K \end{vmatrix} = 0$ the K is equal to ? **[R.U. 2015]**

Sol. : $\begin{vmatrix} 3 & -1 \\ 1 & K \end{vmatrix} = 0$

$\Rightarrow$ $3K + 1 = 0$

$\Rightarrow$ $K = -\frac{1}{3}$

Example 48. Show that for marix $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, $A^{-1} = A^T$ **[R.U. 2015]**

Solution : $$A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$$

$\therefore$ $$A^T = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ 4 & 7 & 4 \end{bmatrix} \quad .....(1)$$

$\because$ $$\text{adj. } A = \frac{1}{81}\begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$|A| = \frac{1}{81}(-8 \times 72 - 1 \times 9 + 4 \times -36) = -81/81 = -1$$

$\therefore$ $$A^{-1} = \frac{1}{-81}\begin{bmatrix} 72 & -36 & -9 \\ -9 & -36 & 72 \\ -36 & -63 & -36 \end{bmatrix}$$

$$= \frac{9}{-81}\begin{bmatrix} 8 & -4 & -1 \\ -1 & -4 & 8 \\ -4 & -7 & -4 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} -8 & 4 & 1 \\ 1 & 4 & -8 \\ -4 & 7 & 4 \end{bmatrix}$$

Hence $A^T = A^{-1}$

Example 49. $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$ **Verify that A. (adi A) = (adj A). A = $|A|I_3$** **[R.U. 2015]**

Solution : $\because$ $$A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$\Rightarrow$ $|A| = 1(-28 + 30) - 3(-21 - 0) + 0(-18-0)$
$= 2 + 63 = 65$

$$\text{adj. } A = \begin{bmatrix} 2 & 21 & 15 \\ 21 & -7 & -5 \\ -18 & 6 & -5 \end{bmatrix}$$

$$\therefore \quad A.\,\text{adj.}\,A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix} \begin{bmatrix} 2 & 21 & 15 \\ 21 & -7 & -5 \\ -18 & 6 & -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2+63+0 & 21-21+0 & 15-15 \\ 6+84-90 & 63-28+30 & 45-20-25 \\ 0-126+126 & 0+42-42 & 0+30+35 \end{bmatrix}$$

$$= \begin{bmatrix} 65 & 0 & 0 \\ 0 & 65 & 0 \\ 0 & 0 & 65 \end{bmatrix}$$

$$= 65 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= 65\, I_3 = |A|\, I_3$$

$$\text{Similarly (adj. A). A} = \begin{bmatrix} 2 & 21 & 15 \\ 21 & -7 & -5 \\ -18 & 6 & -5 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}$$

$$= \begin{bmatrix} 65 & 0 & 0 \\ 0 & 65 & 0 \\ 0 & 0 & 65 \end{bmatrix} = 65\, I_3 = |A|\, I_3$$

$$\therefore \quad A.(\text{adj}.A) = (\text{adj}.A)\, A = |A|\, I_3$$

Example 50. Find the eigen values and corresponding eigen vectors of the matrix ?

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

[R.U. 2015]

Solution : Charactersitic eq. of the give matrix $\Rightarrow |A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0 \quad \Rightarrow \lambda = 0, 3, 15$

$\therefore$ eigen values are $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 15$

Now eigen vector corresponding to $\lambda = 0$

$$\Rightarrow \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - (R_1 + R_2)$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 0 & -5 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0, -x_1 + 7x_2 - 4x_3 = 0$

and $\quad - 5x_2 + 5x_3 = 0$

$\Rightarrow \quad x_2 = x_3$

Hence $\quad 2x_1 - x_2 = 0 \Rightarrow x_2 = 2x_1$

$\therefore \quad$ Let $x_1 = 1 \Rightarrow x_2 = 2, \; x_3 = 2$

$$\therefore \quad B_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ is the required eigen vector.}$$

Now eigen vector corresponding to $\lambda = 3$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & 4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 5x_4 - 6x_5 + 2x_6 = 0$

$-6x_4 + 4x_5 + 4x_6 = 0$

$2x_4 - 4x_5 = 0$

$\Rightarrow \quad x_4 = 2x_5$

Let $\quad x_5 = 1 \Rightarrow x_4 = 2 \Rightarrow x_6 = -2$

$$\therefore \quad B_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \text{ is the required eigen vector.}$$

Now eigen vector coressponding to $\lambda = 15$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1 \Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ 1 & -2 & -6 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2 \Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ 1 & -2 & -6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 + 3R_1 \Rightarrow \begin{bmatrix} -7 & -6 & 2 \\ -20 & -20 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_7 \\ x_8 \\ x_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \quad -20x_7 - 20x_8 = 0 \quad \Rightarrow x_7 = -x_8$

and $-7x_7 - 6x_8 + 2x_9 = 0$

$\Rightarrow \quad x_8 = -x_9$

Let $\quad x_9 = 1 \Rightarrow x_8 = -2$ and $x_7 = 2$

$\therefore \quad B_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda = 15$

Example 51. $A = \begin{bmatrix} 4 & 5 \\ 1 & 3 \end{bmatrix}$ find $A^2 - 4A + 2I_2$ **[R.U. 2016]**

Solution : $A = \begin{bmatrix} 4 & 5 \\ 1 & 3 \end{bmatrix} \Rightarrow A^2 = \begin{bmatrix} 4 & 5 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 21 & 35 \\ 7 & 14 \end{bmatrix}$$

$$\therefore \quad A^2 - 4A + 2I_2 = \begin{bmatrix} 21 & 35 \\ 7 & 14 \end{bmatrix} - \begin{bmatrix} 16 & 20 \\ 4 & 12 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 15 \\ 3 & 4 \end{bmatrix}$$

Example 52. Solve the following system of equations by using **Cramer's rule :**

$$\begin{bmatrix} 3 & 5 & 1 \\ 4 & 2 & 0 \\ 1 & 5 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x \\ 2x+y \\ 4z \end{bmatrix} + \begin{bmatrix} 8 \\ 5 \\ -17 \end{bmatrix}$$

[R.U. 2016]

Solution : Given system of equations is written as

$$3x + 5y + z = 3x + 8 \Rightarrow 5y + z = 8$$
$$4x + 2y = 2x + y + 5 \Rightarrow 2x + y = 5$$
$$x + 5y - 4z = 4z - 17 \Rightarrow x + 5y - 8z = -17$$

$$\therefore \quad D = \begin{vmatrix} 0 & 5 & 1 \\ 2 & 1 & 0 \\ 1 & 5 & -8 \end{vmatrix} = 89$$

$$D_1 = \begin{vmatrix} 8 & 5 & 1 \\ 5 & 1 & 0 \\ -17 & 5 & -8 \end{vmatrix} = 178$$

$$D_2 = \begin{vmatrix} 0 & 8 & 1 \\ 2 & 5 & 0 \\ 1 & -17 & -8 \end{vmatrix} = 89$$

$$D_3 = \begin{vmatrix} 0 & 5 & 8 \\ 2 & 1 & 5 \\ 1 & 5 & -17 \end{vmatrix} = 267$$

$\therefore$ By cramer's rule

$$x = \frac{D_1}{D} = 2$$

$$y = \frac{D_2}{D} = 1$$

$$z = \frac{D_3}{D} = 3$$

Example 53. Find eigens values and eigen vectors for the following matrix

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

[R.U. 2016]

Solution : $\quad A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Let λ be the eigen values of A

Characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \quad \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (6-\lambda)\{(3-\lambda)^2-1\}+2\{-6+2\lambda+2\}+2\{2-6+2\lambda\}=0$

$\Rightarrow (6-\lambda)\{9+\lambda^2-6\lambda-1\}+2\{2\lambda-4\}+2\{2\lambda-4\lambda\}=0$

$\Rightarrow (\lambda^2-6\lambda+8)(6-\lambda)+4(2\lambda-4)=0$

$\Rightarrow 6\lambda^2-36\lambda+48-\lambda^3+6\lambda^2-8\lambda+8\lambda-16=0$

$\Rightarrow -\lambda^3+12\lambda^2-36\lambda+32=0$

$\Rightarrow \lambda^3-12\lambda^2+36\lambda-32=0$

$\Rightarrow (\lambda-2)(\lambda-2)(\lambda-8)=0$

$\Rightarrow \lambda=2,2,8$

$\therefore$ Eigen value of A are 2, 2, and 18

Eigen vectors : Let eigen vectors be X^1, X^2 and X^3

Putting $\lambda = 2$,

$$\begin{bmatrix}4 & -2 & 2\\ -2 & 1 & -1\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\Rightarrow \quad \begin{aligned}4x_1-2x_2+2x_3&=0\\ -3x_1+x_2-x_3&=0\\ 2x_1-x_2+x_3&=0\end{aligned}$$

By cross multipliation

$$\frac{x_1}{0}=\frac{x_2}{0}=\frac{x_3}{0}$$

No eigen vector.

$$\lambda = 8: \begin{bmatrix}-2 & -2 & 2\\ -2 & -5 & -1\\ 2 & -1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\frac{x_1}{12}=\frac{x_2}{6}=\frac{x_3}{6}=K_1$$

$$\Rightarrow \quad X=\begin{bmatrix}2K_1\\ K_1\\ K_1\end{bmatrix}$$

EXERCISES 3.1

1. Give an example of
   (i) a column matrix (ii) a row matrix (iii) a diagonal matrix (iv) a scalar matrix.

2. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$, show that $A^2-5A+7I_2=0$

3. If $A=\begin{bmatrix}2 & -3\\ -7 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 0\\ 2 & -4\end{bmatrix}$, verify that
   (i) $(A+B)^T=A^T+B^T$ (ii) $(2A)^T=2A^T$

4. Find X if $Y=\begin{bmatrix}3 & 2\\ 1 & 4\end{bmatrix}$ and $2X+Y=\begin{bmatrix}1 & 0\\ -3 & 2\end{bmatrix}$

5. If $A=\begin{bmatrix}\cos 2\theta & \sin 2\theta\\ -\sin 2\theta & \cos 2\theta\end{bmatrix}$, find A^2

6. If $X-Y=\begin{bmatrix}1 & 1 & 1\\ 1 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}$ and $X+Y=\begin{bmatrix}3 & 5 & 1\\ -1 & 1 & 4\\ 11 & 8 & 0\end{bmatrix}$, Find X and Y

7. If $2\begin{bmatrix}3 & 4\\ 5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$, Find x and y.

8. If $A=\begin{bmatrix}2 & 3\\ 5 & 7\end{bmatrix}$, $B=\begin{bmatrix}-1 & 0 & 2\\ 3 & 4 & 1\end{bmatrix}$ and $C=\begin{bmatrix}-1 & 2 & 3\\ 2 & 1 & 0\end{bmatrix}$
   Find (i) A + B (ii) B + C

9. If $\begin{bmatrix}x & 3x-y\\ 2x+z & 3y-w\end{bmatrix}=\begin{bmatrix}3 & 2\\ 4 & 7\end{bmatrix}$, find the values of x, y, z and w.

10. If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$, show that $A-A^T$ is a skew symetric matrix.

11. Find the adjoint of each of the following matrices
    (i) $\begin{bmatrix}-3 & 5\\ 2 & 4\end{bmatrix}$ (ii) $\begin{bmatrix}1 & \tan\alpha/2\\ -\tan\alpha/2 & 1\end{bmatrix}$ (iii) $\begin{bmatrix}1 & 2 & 5\\ 2 & 3 & 1\\ -1 & 1 & 1\end{bmatrix}$

12. If $A=\begin{bmatrix}-4 & -3 & -3\\ 1 & 0 & 1\\ 4 & 4 & 3\end{bmatrix}$, show that adj A = A

13. Find the inverse of each of the following matrices

(i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 1 & 4 \\ 1 & 1 & 2 \end{bmatrix}$

14. Let $A = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix}$ $B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$, Find $(AB)^{-1}$

15. Show that for matrix $A = \frac{1}{9}\begin{bmatrix} -8 & 1 & 4 \\ 4 & 4 & 7 \\ 1 & -8 & 4 \end{bmatrix}$, $A^{-1} = A^T$

16. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, find A^{-1} and prove that $A^2 - 4A - 5I = 0$

17. Solve the following system of linear equation by cramer's rule.

(i) $2x - y = 1$, $7x - 2y = 7$

(ii) $9x + 5y = 10$, $3y - 2x = 8$

(iii) $3x + y + z = 2$, $2x - 4y + 3z = -1$, $4x + y - 3z = -11$

18. Solve the following system of equations by matrix method

(i) $x - 2y - 4 = 0$, $-3x + 5y + 7 = 0$

(ii) $x + y + z = 3$, $2x + y + z = 2$, $x - 2y + 3z = 2$

19. Solve the following system of equation by using matrix inverse method

$$3x + 2y + 7 = x + 2z + 21$$
$$x - y + 3z = 3x + 2y - 17$$
$$3x + y - 4 = 2z + y + 10$$

[R.U. 2016]

ANSWERS 3.1

4. $X = \begin{bmatrix} -2 & -2 \\ -4 & -6 \end{bmatrix}$

5. $\begin{bmatrix} \cos 4\theta & \sin 4\theta \\ -\sin 4\theta & \cos 4\theta \end{bmatrix}$

6. $X = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 1 & 2 \\ 6 & 4 & 0 \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \\ 5 & 4 & 0 \end{bmatrix}$

7. $x = 2$ $y = -8$

8. A + B is not defined (ii) $B + C = \begin{bmatrix} -2 & 2 & 5 \\ 5 & 5 & 1 \end{bmatrix}$

9. $x = 3, y = 7, z = -2, w = 14$

11. (i) $\begin{bmatrix} 4 & -5 \\ -2 & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & \tan\alpha/2 \\ \tan\alpha/2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$

13. (i) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (ii) $\frac{1}{17}\begin{bmatrix} 1 & -5 \\ 3 & 2 \end{bmatrix}$ (iii) $\frac{1}{4}\begin{bmatrix} -2 & -1 & 3 \\ -2 & -3 & 11 \\ 2 & 1 & -5 \end{bmatrix}$

14. $\frac{1}{271}\begin{bmatrix} 94 & -39 \\ -75 & 34 \end{bmatrix}$

17. (i) $x = \frac{5}{3}, y = \frac{7}{3}$

(ii) $x = \frac{-10}{17}, y = \frac{52}{17}$

(iii) $x = -1,\ y = 2,\ z = 3$