Chapter 1

# Functions

## 1.1 Introduction

Calculus is that branch of Mathematics, in which we study the variable quantities. It has two branches : Differential Calculus and Integral Calculus. In differential calculus we study the problems related with the rate of change of quantities.

## 1.2 Quantities

(i) **Variable :–** A quantity used in a mathematical operation which can take any value from a particular set is called variable quantity. It is denoted by $x, y, z$.

(ii) **Constant :–** A quantity whose value does not change in a mathematical operation is called constant. It is denoted as $a, b, c$.

(iii) **Interval :–** The set of all real numbers lying between two real numbers $a$ and $b$ $(a < b)$ is called an interval. Length of the interval is $b - a$.

(a) **Open interval :–** The interval in which terminal points $a$ and $b$ are not included is called open interval. It is denoted as $(a, b)$ and is equal to $\{x : a < x < b\}$.

(b) **Closed interval :–** The interval in which terminal point $a$ and $b$ are included is called closed interval. It is denoted as $[a, b]$ and is equal to $\{x : a \le x \le b\}$.

(c) **Semi closed and semi open interval :–** That interval in which one terminal point is not included is known as semi closed and semi open interval. It is denoted as $[a, b) = \{x : a \le x < b\}$ (left closed and right open) or $(a, b] = \{x : a < x \le b\}$

## 1.3 Function

A function defined from a set A to a set B is a rule which associates each element of set A (say $x$) to some unique element of set B (say $f(x)$ and it is written as $f : A \to B$.

If an element $a \in$ A is related to some element b $\in$ B under the function $f$ then it is written as $b = f(a)$. '$b$' is said to be f-image of '$a$' and '$a$' is said to be pre image of 'b'.

**Domain, Co-domain and Range of a function**

If $f : A \to B$ be a function then A is called domain, B is called co-domain of $f$ and the set of $f$-images of elements of A is called range of $f$.

Range = $\{f(a) : f(a) \in B, \forall\, a \in A\}$

**Defined and Undefined function :-** If $f(x)$ be a function of $x$, at $x = a$ the value of the function $f(a)$ be a real number then function is said to be defined at $x = a$ and if $f(a)$ is not a real number then f is said to be undefined at $x = a$.

**Example :** The function $f(x) = \sqrt{4-x^2}$ will be defined if $-2 \le x \le 2$.

**Even and Odd function** A function $f(x)$ is said to be an even function if for all values of $x$ :

$$f(-x) = f(x)$$

and an odd function if $f(-x) = -f(x)$

## Illustrative Examples

◻ **Example 1. If $f(x) = \sqrt{25-x^2}$ then find the value of $f(3), f(-4)$ and domain of $f$.**

**Solution:**

$$f(3) = \sqrt{25-(3)^2} = 4$$

$\because$ $f(-4) = \sqrt{25-(-4)^2} = 3$

$\because$ $f$ is defined if $25 - x^2 \ge 0$

$\Rightarrow$ $x^2 \le 25$

$\Rightarrow$ $x \in [-5, 5]$

$\therefore$ Domain of $f$ $= [-5, 5]$

◻ **Example 2. For what values of $x$ the function $f(x) = \sqrt{\log\left(\frac{5x-x^2}{6}\right)}$ will be defined ?**

**Solution:** $f$ will be defined if $\log\left(\frac{5x-x^2}{6}\right) \ge 0 \Rightarrow \frac{5x-x^2}{6} \ge 1$

$\therefore$ $5x - x^2 \ge 6$

$\Rightarrow$ $x^2 - 5x + 6 \le 0$

$\Rightarrow$ $(x-2)(x-3) \le 0$

$\therefore$ $x \in [2, 3]$

◻ **Example 3. Find the range of the function $f(x) = \frac{x}{1+x^2}$.**

**Solution:** Let $f(x) = y$



$\Rightarrow$ $\frac{x}{1+x^2} = y$

$\Rightarrow x^2y - x + y = 0$

For real value of $x$,

$1 - 4y^2 \ge 0$

$\Rightarrow$ $(1-2y)(1+2y) \ge 0$

$\Rightarrow$ $\frac{-1}{2} \le y \le \frac{1}{2}$

$\therefore$ Range = $\left[-\frac{1}{2}, \frac{1}{2}\right]$

◻ **Example 4. If $f(x) = e^x$ then prove that**

$$f(x+y) = f(x) \cdot f(y)$$

**Solution:** $\therefore$ $f(x) = e^x$

$$f(x+y) = e^{x+y} = e^x \cdot e^y$$
$$= f(x) \cdot f(y)$$

◻ **Example 5. If $f(x) = \begin{cases} 3x-1 , & x > 3 \\ x^2 - 2 , & -2 \le x \le 3 \\ 2x+3 , & x < -2 \end{cases}$ then find the value of $f(2)$.**

**Solution:** $\because$ $2 \in [-2, 3]$

$\therefore$ Using $f(x) = x^2 - 2$

$$f(2) = (2)^2 - 2 = 2$$

◻ **Example 6. Find the domain and range of the following function**

$$f(x) = \frac{ax+b}{cx-d}$$ **[R.U. 2015]**

**Solution: Domain :** $f(x)$ will be undefined if $cx - d = 0$ or $x = d/c$ so domain of $f = R - \{d/c\}$

**Range :** If $f(x) = y$

$\Rightarrow$ $\frac{ax+b}{cx-d} = y$ or $ax + b = y(cx - d)$

or $x(cy - a) = dy + b$

or $x = \frac{dy+b}{cy-a}$

$x$ will be undefined if $cy - a = 0$ or $y = a/c$

$\therefore$ Range of $f = R - \{a/c\}$

## EXERCISE 1.1

1. If $f(x) = \log \frac{1+x}{1-x}$ then show that

$$f(a) + f(b) = f\left(\frac{a+b}{1+ab}\right)$$

2. If $f(x) = \frac{1-x^2}{1+x^2}$, then show that

$$f(\tan\theta) = \cos 2\theta$$

3. If $y = f(x) = \frac{5x+3}{4x-5}$, then show that

$$x = f(y)$$

4. If $f(x) = \log\left(\frac{x}{x-1}\right)$, then show that

$$f(x+1) + f(x) = \log\left(\frac{x+1}{x-1}\right)$$

5. Find whether the following functions are even or odd :

(i) $f(x) = x^3 - 4x - \frac{4}{x} + \frac{1}{x^3}$ (ii) $f(x) = \log\left(x + \sqrt{x^2+1}\right)$

(iii) $f(x) = x\left(\frac{a^x - 1}{a^x + 1}\right)$ (iv) $f(x) = \sin x + \cos x$

6. Find the domain of :

(i) $f(x) = \frac{x}{3x-4}$ (ii) $f(x) = \frac{1}{\sqrt{4-x^2}}$

(iii) $f(x) = \sqrt{x - \sqrt{1-x^2}}$ (iv) $f(x) = \log_e\left(\sqrt{x-4} + \sqrt{6-x}\right)$

7. Find the range of :

(i) $f(x) = \frac{2+x}{2-x}$ (ii) $f(x) = \frac{x^2-x+1}{x^2+x+1}$ (iii) $f(x) = e^x + e^{-x}$

## ANSWERS 1.1

5. (i) odd (ii) odd
(iii) even (iv) Niehter odd nor even

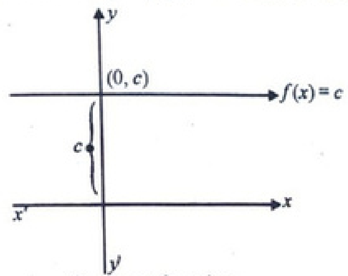
6. (i) $R - \left\{\frac{4}{3}\right\}$ (ii) [4, 6]

7. (i) R – {– 1} (ii) $\left[\frac{1}{3}, 3\right]$
(iii) $(-\infty, \infty)$



## 1.4 Graphs of Real Functions [R.U. 2016]

**(1) Constant function :–** If range of a function is constnt then it is called a constant function such as $y = f(x) = c$ where $c$ is constant.




Constant function
**Fig. 1.1**

**(2) Identity function :–** The function $f(x) = x$, $\forall x \in R$ is called identity function.

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | –2 | –1 | 0 | 1 | 2 |

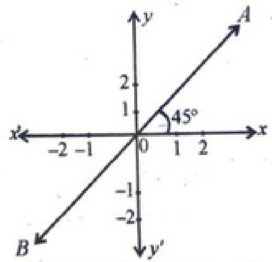



**Fig. 1.2 Identity function**

**(3) Modulus function :–** The function defined as

$$f(x) = |x| = \begin{cases} x & \text{when } x \ge 0 \\ -x & \text{when } x < 0 \end{cases}$$ is

called modulus function.

| $x$ | –2 | –1 | 0 | 1 | 2 |
|---|---|---|---|---|---|
| $f(x)$ | 2 | 1 | 0 | 1 | 2 |

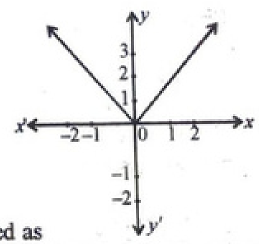



**Fig. 1.3 : Modulus function**

**(4) Signum function :–** The function defined as

$$f(x) = \begin{cases} \frac{|x|}{x}, & \text{when } x \ne 0 \\ 0, & \text{when } x = 0 \end{cases}$$

$$= \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$$

is called Signum function.

| $x$ | –3 | –2 | –1 | 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|---|---|
| $f(x)$ | –1 | –1 | –1 | 0 | 1 | 1 | 1 |

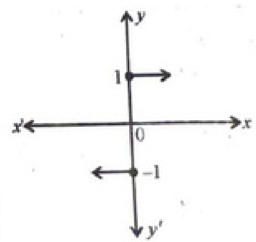



**Fig. 1.4 : Signum function**

**(5) Greatest Integer function :–** The function defined as $[x]$ = greatest integer $< x$ is called the greater integer function.

**For example** $\left[-\frac{5}{3}\right] = -2$ and $\left[\frac{5}{3}\right] = 1$

Note : (i) $[x] \le x$ (ii) $[x] + 1 > x$

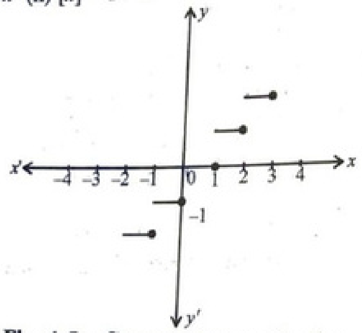



Fig. 1.5 : Greatest Integer function

**(6) Trigonometric function :–**

**(1) $f(x) = \sin x$**

Domain = $R$ and Range = $[-1, 1]$

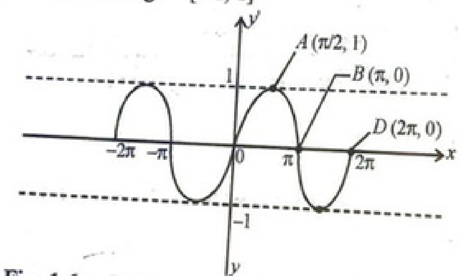



Fig. 1.6 : Trigonometric function $f(x) = \sin x$

**(2) $f(x) = \cos x$**

Domain = $R$ and Range = $[-1, 1]$

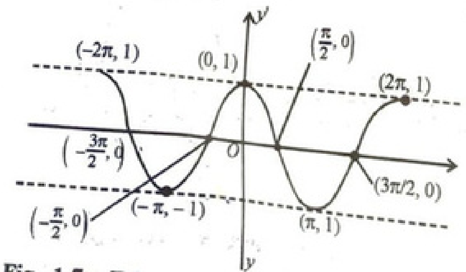



Fig.–1.7 : Trignometric function $f(x) = \cos x$



**(3) $f(x) = \tan x$**

Domain = $R - \left\{\frac{(2n+1)\pi}{2}, n \in Z\right\}$

and Range = $R$

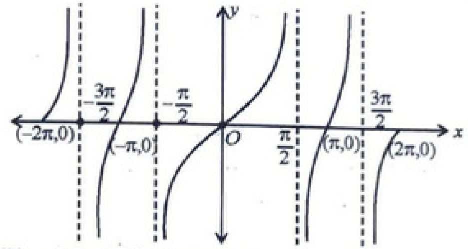



Fig.–1.8 : Trignometric function $f(x) = \tan x$

**(4) $f(x) = \operatorname{cosec} x$**

Domain = $R - \left\{\frac{n\pi}{2} / n \in Z\right\}$ and Range = $R - (-1, 1)$

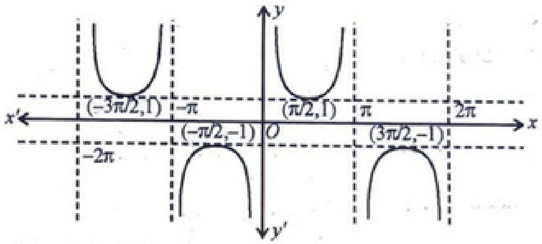



Fig.–1.9 : Trignometric function $f(x) = \operatorname{cosec} x$

**(5) $f(x) = \sec x$**

Domain = $R - \left\{\frac{(2n+1)}{2} / n \in Z\right\}$ and Range = $R = (-1, 1)$

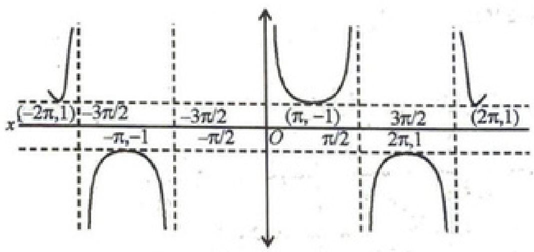



Fig.–1.10 : Trignometric function $f(x) = \sec x$

**(6) $f(x) = \cot x$**

Domain $= R - \{n\pi \,/ n \in Z\}$

and Range $= R$

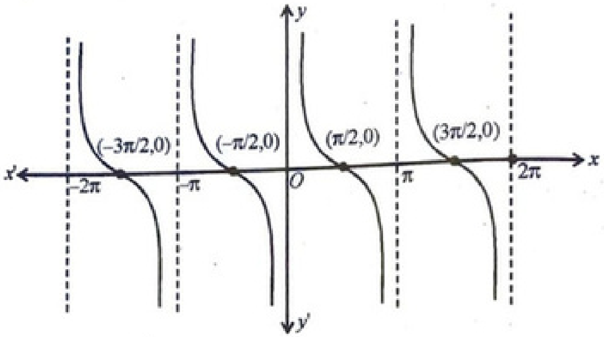



**Fig.–1.11 : Trignometric function $f(x) = \cot x$**

**(7) Inverse Trignometrical functions**

**(1) $f(x) = \sin^{-1} x$**

Domain $= [-1, 1]$

and Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

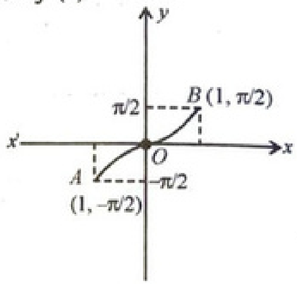



**Fig. 14.12 : Inverse Trignometrical function $f(x) = \sin^{-1} x$**

**(2) $f(x) = \cos^{-1} x$**

Domain $= [-1, 1]$

and Range $= [0, \pi]$

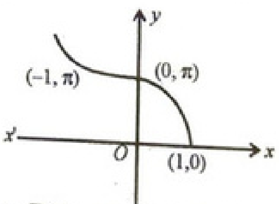



**Fig.1.13 : Inverse Trignometrical function $f(x) = \cos^{-1} x$**

**(3) $f(x) = \tan^{-1} x$**

Domain $= R$

and Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



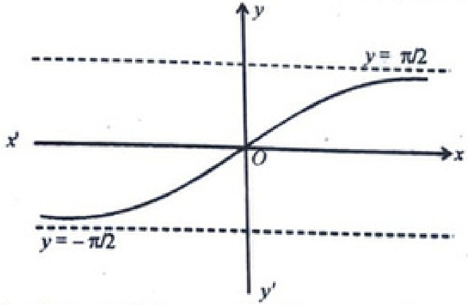



**Fig.1.14 : Inverse Trignometrical function $f(x) = \tan^{-1} x$**

**(4) $f(x) = \cot^{-1} x$**

Domain $= R$

and Range $= (0, \pi)$

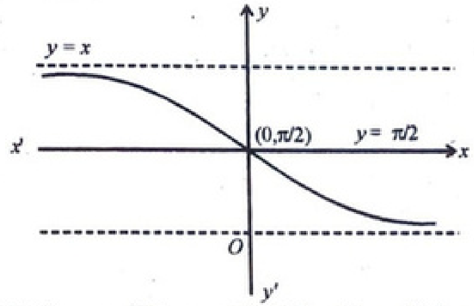



**Fig. 1.15 :Inverse Trignometrical function $f(x) = \cot^{-1} x$**

**(5) $f(x) = \mathrm{cosec}^{-1} x$**

Domain $= R - (-1, 1)$

and Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

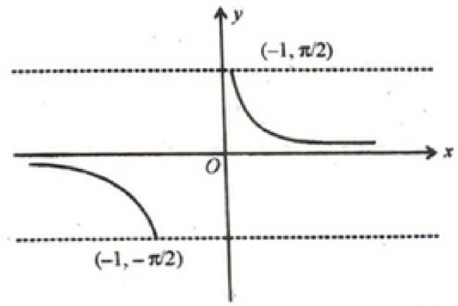



**Fig.1.16 : Inverse Trignometrical function $f(x) = \mathrm{cosec}^{-1} x$**

(6) $f(x) = \sec^{-1} x$

Domain $= R - (-1, 1)$

and Range $= [0, \pi] - \{\pi/2\}$

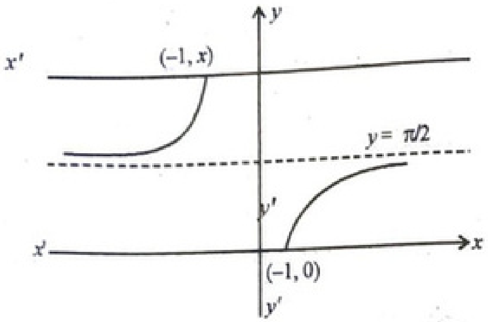



**Fig.1.17 : Inverse Trignometrical function $f(x) = \sec^{-1} x$**

**(8) Hyperbolic Functions**

(1) $f(x) = \sin hx$

Domain $= (-\infty, \infty)$

and Range $= (-\infty, \infty)$

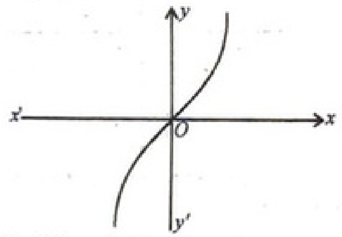



**Fig. 1.18 : Hyperbolic Function $f(x) = \sin hx$**

(2) $f(x) = \cos hx$

Domain $= (-\infty, \infty)$

and Range $= [1, \infty)$

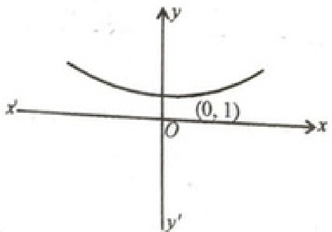



**Fig.1.19 : Hyperbolic Function $f(x) = \cos hx$**

(3) $f(x) = \tan hx$

Domain $= (-\infty, \infty)$

and Range $= (-1, 1)$



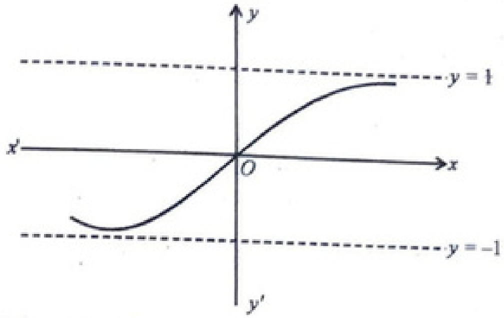



**Fig. 1.20 : Hyperbolic Function $f(x) = \tan hx$**

**(9) Logarithmic Functions**

(1) $f(x) = \log_a x$ $(a > 1)$

Domain $= R^+$

and Range $= R$

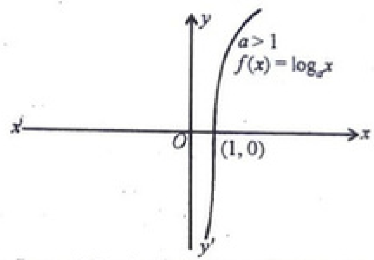



**Fig.–1.21 : Logarithmic Function $f(x) = \log_a x$ $(a > 1)$**

(2) $f(x) = \log_a x$ $(a < 1)$

Domain $= R^+$

and Range $= R$

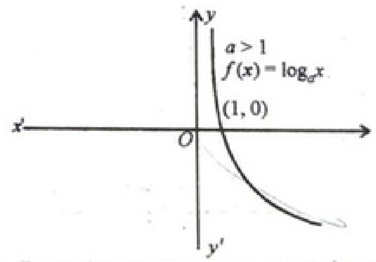



**Fig.1.22 : Logarithmic Function $f(x) = \log_a x$ $(a < 1)$**

**(10) Exponential Functions**

(1) $f(x) = a^x$, $(a > 1)$

Domain $= R$

Range $= R^+$

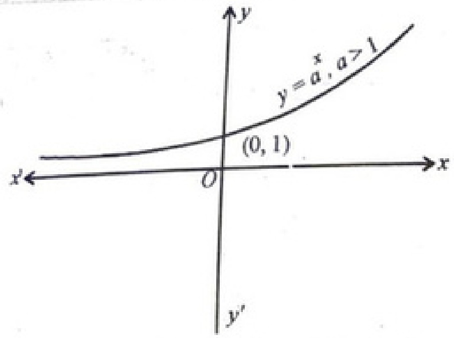



**Fig.1.23 : Exponential Function $f(x) = a^x$ $(a > 1)$**

**(2) $f(x) = a^x$, $(0 < a < 1)$**

Domain $= R$

Range $= R^+$

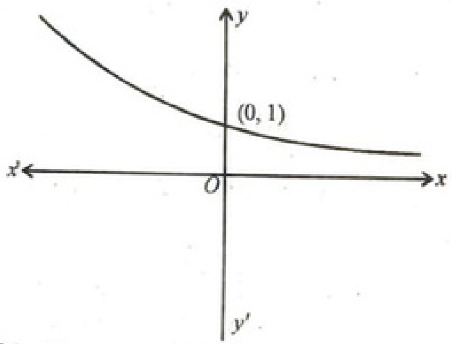



**Fig. 1.24 : Exponential Function $f(x) = a^x$ $(0 < a < 1)$**

## Illustrative Examples

**Example 7 :** Draw the graph of the function $y = |x - 2| + |x - 3|$ in the interval $[-4, 4]$.

**Solution :**

$$y = \begin{cases} -x+2-x+3; & -4 \le x < 2 \\ x-2-x+3; & 2 \le x < 3 \\ x-2+x-3; & 3 \le x \le 4 \end{cases}$$

$$= \begin{cases} 5-2x; & -4 \le x < 2 \\ 1; & 2 \le x < 3 \\ 2x-5; & 3 \le x \le 4 \end{cases}$$

Tabulating the values of $x$ and $y$.



| $x$ | $-4$ | $-3$ | $-2$ | $-1$ | 0 | 1 | 2 | 2·5 | 3 | 3·5 | 4 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $y$ | 13 | 11 | 9 | 7 | 5 | 3 | 1 | 1 | 1 | 2 | 3 |

Following is the graph

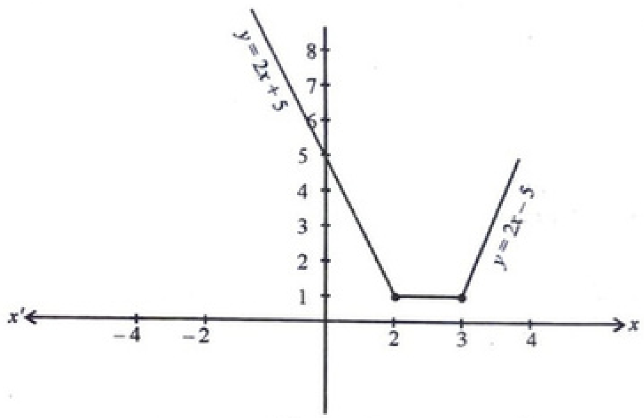



**Fig. 1.25**

**Example 8 : Draw the graph of $f(x) = \cos\frac{x}{2}$ in the interval $[-2\pi, 2\pi]$.**

**Solution :** Tabulating the values of $x$ and $y$

| $x$ | 0 | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $2\pi$ | $-\frac{2\pi}{3}$ | $-\pi$ | $-\frac{4\pi}{3}$ | $-2\pi$ |
|---|---|---|---|---|---|---|---|---|---|
| $f(x)$ | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-1$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-1$ |

Following graph is obtained

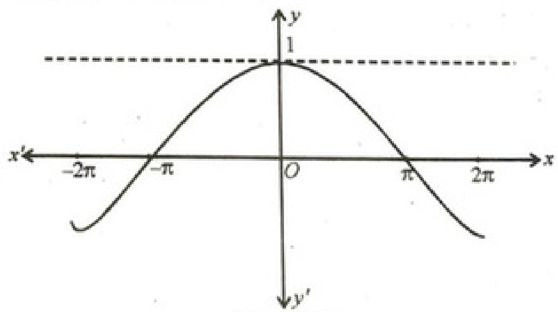


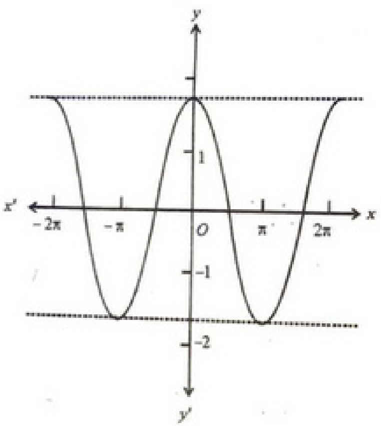

**Fig. 1.26**

## EXERCISES 1.2

1. Draw the graph of function $y = 2\cos x$ when $x \in [-2\pi, 2\pi]$
2. Draw the graph of function $y = 3\cos 2x$ when $x \in [-\pi, \pi]$
3. Draw the graph of function $f(x) = -\sqrt{4-x^2}$
4. Draw the graph of function $f(x) = \begin{cases} -x, & -4 \le x < 0 \\ x, & 0 \le x \le 1 \\ 2-x, & 1 < x \le 4 \end{cases}$ in the interval $[-4, 4]$.
5. Draw the graph of function $f(x) = |x| + |x-1|$ when $-3 \le x \le 3$

## ANSWERS 1.2

1.

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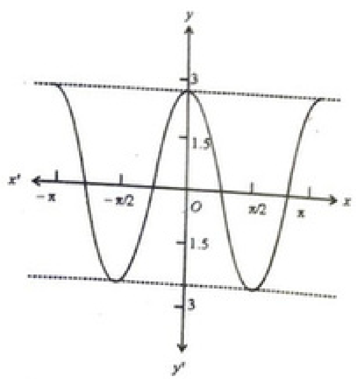


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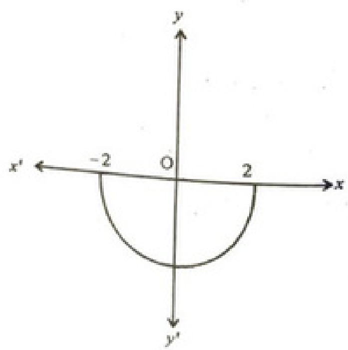




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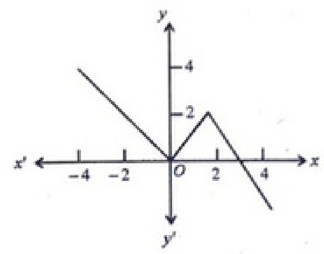
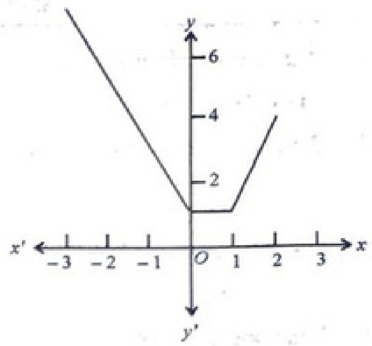



5.



Chapter 2

# Functions and Binary Operations

## 2.1 Functions

Let $A$ and $B$ be two non empty sets. Then a rule or a correspondence $f$ which associates to each element $x \in A$, a unique element denoted by $f(x)$ of $B$ is called a function or a mapping from $A$ to $B$ and we write

$$f : A \to B$$

The element $f(x)$ of $B$ is called the image of $x$ under $f$ while $x$ is called pre image of $f(x)$ under $f$.

In other words a relation $f$ from a set $A$ to a set $B$ is said to be a function if every element of set $A$ has one and only one image in set $B$.

## 2.2 Domain, Co-domain and Range of Functions :

Let $f : A \to B$, then the set $A$ is known as the domain of $f$ and $B$ is known as the co-domain of $f$. The set $f$ of all $f$-images of elements of $A$ is known as the image of $f$ or image set of $A$ under $f$ and is denoted by $f(A)$.

Thus, $f(A) = \{f(x) : x \in A\}$ = range of $f$

clearly $f(A) \subseteq B : x \in A$ = co-domain of $f$

## 2.3 Various type of Functions

- **Many one function :** Let $f : A \to B$ If two or more than two elements of $A$ have the same image in $B$. Then $f$ is said to be many one.
- **One-one function (injection) :** Let $f : A \to B$ then $f$ is said to be one-one function or an injection if different elements of $A$ have different images in $B$.

  Thus $f : A \to B$ is one one $\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ $\forall\ a, b \in A$. $\Leftrightarrow$
  $f(a) = f(b) \Rightarrow a = b$ $\forall\ a, b \in A$.
- **Onto function (Surjection) :** Let $f : A \to B$ if every element in $B$ has at least one pre-image in $A$ then $f$ is said to be an onto function.



Thus $f : A \to B$ is a surjection iff for each $b \in B\ \exists\ a \in A$ such that $f(a) = b$ clearly, $f$ is onto $\Leftrightarrow$ range $(f) = B$ = codomain $f$.

- **Into function :** Let $f : A \to B$ if there exists even a single element in $B$ having no. preimage in $A$. Then $f$ is said to be an into function
- **Bijective Function :-** A one-one and onto function is said to be bijective. A bijective function is also known as a one to one correspondence.

  In other words, a function $f : A \to B$ is a bijective if (i) it is one-one i.e. $f(x) = f(y) \Rightarrow x = y\ \forall\ x, y \in A$

  (ii) it is onto i.e. $\forall\ y \in B$, there exists $x \in A$ such that $f(x) = y$.
- **Constant function :** Let $f : A \to B$ defined in such a way that all the element in $A$ have the same image in $B$. then $f$ is said to be a constant function.
- **Identity function :** Let A be a non empty set then the function $I_A$, defined by $I_A : A \to A : I_A(x) = x\ \forall\ x \in A$ is called an identity function on $A$. This is clearly a one-one onto function with domain $A$ and range $A$.
- **Equal Function :** Two functions $f$ and $g$ are said to be equal, written as $f = g$ if they have the same domain and they satisfy the condition $f(x) = g(x)\ \forall\ x$ in domain of $f$(or $g$)

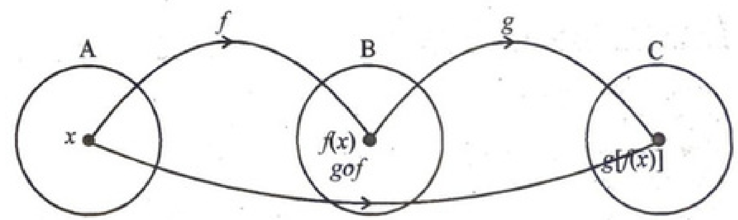
## 2.4 Composite of Functions & Invertible Function :

Let $f : A \to B$ and $g : B \to C$ be two function then composite of the functions $f$ and $g$ denoted by $gof$ is a function from $A$ to $C$ given by

$$gof : A \to C\ (gof)\ (x) = g[f(x)]$$

obviously $gof$ is defined if

range of $f \subset$ domain of $g$.




A function $f : X \to Y$ is set to be Invertible if another function $g : Y \to X$ exists such that $gof = I_X$ and $fog = I_Y$. Function $g$ is called inverse of $f$ and it is denoted as $f^{-1}$.

A function $f$ is invertible if and only if it is one-one and onto.

**Note 1.** $gof$ is composite of $f$ and $g$ whereas $fog$ is composite of $g$ and $f$.

2. $gof$ is undefined if $R_f \cap D_g = \phi$ or $gof$ is defined if $R_f \cap D_g \neq \phi$ or $R_f \subseteq D_g$.
3. $D_{gof} = D_f$ if $D_g = R$
4. $D_{gof} = D_f$ if $R_f \subset D_g$
5. In general $D_{gof} = \{x \in D_f \text{ and } f(x) \in D_g\}$
6. The existence of $fog$ and $fog$ is independent of each other i.e. if $fog$ exists then $gof$ may or may not exist and vice versa.
7. In general $gof \neq fog$
8. $(fog)\ oh = fo\ (goh)$ (Associative law)
9. If $f : R \to R$, $g : R \to R$ then $fog$, $gof$, $gog$ are all defined from $R \to R$
10. If $gof$ is one-one then $f$ is one-one and if $gof$ is onto then $g$ is onto.

## 2.5 Binary Operations :

**Definition. :** An operation over a set is a rule which combines any two elements of the set.

An operation 'O' is called a binary operation on set $A$ if $aob \in A \quad \forall\ a, b \in A$

## 2.6 Types of Binary Operations

Let $A$ be a non empty set and 'O' be a binary operation defined on $A$.

**(i) Associative binary operation :** The binary operation '$o$' on a set $A$ is said to be associative, if $(aob)oc = ao(boc)$, $\forall\, a, b, c \in A$.

**(ii) Commutative Binary operation :** The binary operation '$o$' on a set $A$ is said to be commulative if $aob = boa\ \forall\ a, b \in A$

**(iii) Identity element of set w.r. to binary operation :** An element $b \in A$ is said to be an inverse of element a of set $A$. w.r.t the binary operation '$o$'

if $aob = e = boa$

where $e$ is an identity element of set $A$.

**Theorem 1 :-** Prove that the identity element of set $A$. w.r.t the binary operation 'O' if it exist is unique.

**Proof :** If possible suppose there exist two identity elements $e_1$ and $e_2$ of set $A$ w.r.t. binary operation '$o$'.

Now $e_1 \in A$ and $e_2$ is identity element of $A$

$\Rightarrow \quad e_1\, o\, e_2 = e_1 = e_2\, o\, e_1 \quad$ ....(i)

Also $e_2 \in A$ and $e_1$ is identity element of $A$



$\Rightarrow \quad e_2\, o\, e_1 = e_2 = e_1\, o\, e_2 \quad$ ....(ii)

$\therefore \quad e_1 = e_1\, o\, e_2 = e_2 \quad$ [using (i) and (ii)]

So, identity element of set $A$ w.r.t '$o$' is unique

**Theorem 2 :** Prove that inverse of an element a of set $A$ w.r.t. the associative binary operation '$o$' with identity element $e$ is unique.

**Proof :** If possible suppose $a \in A$ has two inverses $b$ and $b'$

$b$ is inverse of a w.r.t '$o$' $\Rightarrow aob = e = boa$ (def.) ....(1)

$b'$ is inverse of a w.r.t '$o$' $\Rightarrow aob' = e = b'oa$ (def.) ....(2)

Now,
$$b = boe = bo\,(aob') \quad \text{[using (2)]}$$
$$= (boa)\,ob' \quad [\because \text{'}o\text{' is associative}]$$
$$= eob' \quad \text{[using (1)]}$$
$$= b'$$

The inverse of $a$ is unique.

**Examples of Binary Operations :**

(i) Addition is a binary operation on the sets $N, Z, Q, R, C$

$N$ = Set of natural number
$Z$ = Set of integers
$Q$ = Set of all rational numbers
$R$ = Set of all real numbers
$C$ = Set of all complex numbers

Reason, let $a, b \in N$

Then $a + b \in N$

Thus addition is a binary operation on $N$, Z, Q, $R$, $C$ each. Hence $N$, Z, Q, $R$,$C$ each is closed w.r.t. addition operation '+'.

(ii) Subtraction is not a binary operation on $N$, because $3, 7 \in N$ but $3 - 7 \in N$.

(iii) Subtraction is a binary operation on Z, Q, $R$, $C$ each.

(iv) Multiplication is a binary operation on $N$, Z, Q, $R$, $C$ each

(v) Division is a binary operation on $Q - \{0\}$, $R - \{0\}$

(vi) Division is not a binary operation on $N, Z, Q, R, C$.

### Illustrative Examples

■ **Example 1. Let $A$ and $B$ be two sets, show that $f : A \times B \to B \times A$ such that $f(a, b) = (b, a)$ is a bijective function**

**Sol.** Injective ie. one-one

Let $(a_1, b_1)$ and $(a_2\ b_2) \in A \times B$ such that

$f(a_1, b_1) = f(a_2\ b_2)$

$f(a_1, b_1) = f(a_2\ b_2)$

$(b_1, a_1) = (b_2, a_2) \Rightarrow b_1 = b_2$ and $a_1 = a_2$

$(a_1, b_1) = (a_2\ b_2)$

$f(a_1, b_1) = f(a_2\, b_2) \Rightarrow (a_1\, b_1) = (a_2, b_2)\ \forall\ (a_1, b_1)\ (a_2\, b_2) \in A \times B$

Hence $f$ is an injective map i.e. $f$ is one-one

Surjective i.e. onto :

Let $(b, a)$ be any arbitrary element of $B \times A$

$\therefore\ b \in B$ and $a \in A \Rightarrow (a, b) \in A \times B$

So $\forall\ (b, a) \in B \times A,\ \exists\ (a, b) \in A \times B$ such that $f(a, b) = (b, a)$

$\because f : A \to B \to B \times A$ is an onto function.

Hence $f$ is a bijective function i.e. bijection.

■ **Example 2 Let $A = R - \{2\}$ and $B = R - \{1\}$ if $f : A \to B$ is a mapping defined by $f(x) = \dfrac{x-1}{x-2}$ show that $f$ is bijective**

**Sol. Injectivity :** Let $x, y$ be any two elements of $A$

$$f(x) = f(y) \Rightarrow \frac{x-1}{x-2} = \frac{y-1}{y-2}$$

$\Rightarrow (x - 1)(y - 2) = (x - 2)(y - 1)$

$\Rightarrow xy - y - 2x + 2 = xy - x - 2y + 2 \Rightarrow x = y$

Thus $f(x) = f(y) \Rightarrow x = y\ \ \forall\ x, y \in A$ so, $f$ is an injective map.

Surjectivity : Let y be an arbitrary element of $B$, then $f(x) = y \Rightarrow \dfrac{x-1}{x-2} = y$

$\Rightarrow$ $$x - 1 = y(x - 2) \Rightarrow x = \frac{1-2y}{1-y}$$

clearly $$x = \frac{1-2y}{1-y} \text{ is a real number } \forall\ y \neq 1$$

Also $$x = \frac{1-2y}{1-y} \neq 2 \text{ for any } y,$$

for if we take $\dfrac{1-2y}{1-y} = 2$ then we get $1 = 0$ which is wrong. Thus every element y in $B$ has its pre image x in $A$ given by

$$x = \frac{1-2y}{1-y} \text{ and } f(x) = \left(\frac{1-2y}{1-y}\right)$$

$$= \left(\frac{\frac{1-2y}{1-y} - 1}{\frac{1-2y}{1-y} - 2}\right) = \frac{-y}{-1} = y \text{ so } f \text{ is}$$

a surjective map.

Hence $f$ is a bijective map ie. both one-one and onto.



■ **Example 3. Determine which of the following function $f : R \to R$ are (a) one to one (b) onto**

**(i) $f(x) = x + 1$** **(ii) $f(x) = x^3$**

**(iii) $f(x) = |x| + x$** **(iv)** $fx = \begin{cases} 1 & if\ x - \text{is rational} \\ -1 & if\ x - \text{irrational} \end{cases}$

**Note : For onto functions range $(f)$ = co-domain $(f)$**

**Sol.** (i) $f(x) = x + 1$

(a) Injectivity i.e. one to one

Let $x$ and $y$ be two arbitrary elements of domain

$(f) = R$ such that $= f(x) = f(y) \Rightarrow x + 1 = y + 1 \Rightarrow x = y$

$\therefore f$ is one-one function.

(b) subjectivity ie onto

Let $y$ be an element of co-domain (f) $= R$

Now $y = f(x) = x + 1 \Rightarrow x = y - 1 \in R$

$\therefore\ \forall\ y \in$ co domain $(f) = R\ \exists$

$x = y - 1$ in domain (f) $= R$ such that

$$f(x) = f(y - 1) = (y - 1) + 1$$
$$f(x) = y$$

Hence $f : R \to R$ is an onto function.

(ii) $f(x) = x^3$.

(a) Injectivity ie one to one

Let $x, y$ be two arbitrary elements of $R$ = domain $(f)$ such that

$$f(x) = f(y) \Rightarrow x^3 = y^3 \Rightarrow x = y$$

$\therefore f$ is one-one function

(b) Subjectivity ie onto

Let $y$ be any element of co-domain $(f) = R$

Now $$y = f(x) = x^3 \Rightarrow x = y^{1/3}$$

$\therefore\ \forall\ y \in$ codomain $(f) = R\ \exists$

$x = y^{1/3}$ in domain $f = R$ such that

$$f(x) = f(y^{1/3}) = (y^{1/3})^3$$
$$f(x) = y \quad \text{[from (1)]}$$

Hence $f : R \to R$ is an onto function

(iii) $$f(x) = x + |x| = \begin{cases} x + x = 2x & if\ x \geq 0 \\ -x + x = 0 & if\ x < 0 \end{cases} \quad ....(i)$$

Note that by Def.

$$|x| = x \text{ when } x \geq 0$$
$$|x| = -x \text{ when } x < 0$$

(a) Injectivity i.e. one one

Let $x$ and $y$ be any two negative elements of domain $(f) = R$

$$f(x) = 0 \quad \text{and} \quad f(y) = 0$$
$$f(x) = f(y) \quad \text{and} \quad x \neq y$$

Hence $f$ is not one-one function

(b) Surjectivity : i.e. onto

from (1) $$y = f(x) = \begin{cases} 2x & if \ x \geq 0 \\ 0 & if \ x < 0 \end{cases}$$

Clearly range $(f)$ = set of all non-negative real numbers

$[\because y = 2x, x \geq 0 \Rightarrow y \geq 0]$

$\therefore$ Range $(f) \subseteq$ co-domain $(f)$ i.e. $R$

$\therefore$ Range $(f) \neq$ co-domain $(f)$

Hence $f$ is not onto

(iv) $$f(x) = \begin{cases} 1 & if \ x \text{ is rational} \\ -1 & if \ x \text{ is irrational} \end{cases}$$

(a) Injectivity ie one-one

Let $x$, $y$ be any two rational elements of domain $(f)$

$\therefore$ $f(x) = 1$ $f(y) = 1$

i.e. $f(x) = f(y)$ and $x \neq y$

$\therefore$ $f$ is not one-one

(b) Surjectivity ie onto

Here Range $(f) = \{-1, 1\} \neq R$ the co-domain $(f)$ hence $f$ is not onto.

■ **Example 4. Let $f : \{2, 3, 4, 5\} \to \{3, 4, 5, 9\}$ and $g : \{3, 4, 5, 9\} \to \{7, 11, 15\}$ be function defined as $f(2) = 3$, $f(3) = 4$, $f(4) = 5$, $f(5)=5$ and $g(3) = g(4) = 7$ and $g(5) = g(9) = 11$ find $gof$**

**Sol.** Here $R_f = \{3, 4, 5\}$ $CD_g$ $\Rightarrow$ $gof$ is defined

we have $gof(2) = g[f(2)] = g(3) = 7$ $gof(3) = g(f(3)) = g(4) = 7$

$gof(4) = g(f(4)) = g(5) = 11$ and $gof(5) = gf(5) = g(5) = 11$,

$gof = \{(2, 7), (3,7) (4, 11) (5, 11)\}$

■ **Example 5. If $f(x) = x^2 + x + 1$ and $g(x) = \sin x$ then show that $fog$ and $gof$ are not equal.**

**Sol.** $f(x) = x^2 + x + 1$, $D_f = R$, $R_f = \left[\frac{3}{4}, \infty\right]$

$[\because f(x)$ is defined $\forall \ x \in R$ and $-1 \leq \sin x \leq 1 \ \forall \ x]$

$R_g \cap D_f = [-1, 1] \neq \phi$ $fog$ is defined

$fog(x) = f[g(x)] = f(\sin x) = \sin^2 x + \sin x + 1$

$R_g \subset D_f \Rightarrow D_{fog} = D_g = R$



$$R_f \cap D_g = \left[\frac{3}{4}, \infty\right) \neq \phi \Rightarrow gof \text{ is defined}$$

$\therefore$ $gof(x) = g[f(x)] = g(x^2 + x + 1) = \sin(x^2 + x + 1)$

$R_f \subset D_g \Rightarrow D_{gof} = D_f = R$

Clearly $fog \neq gof$

■ **Example 6. If $f : R \to R$, $f(x) = x^2$ then find the**

**(i) range of $f$**

**(ii) $\{x \mid f(x) = 4\}$**

**(iii) $\{y \mid f(y) = -1\}$**

**Sol.** (i) Range of $f = \{x \in R / 0 < x < \infty\}$ **Ans.**

(ii) $\{x \mid f(x) = 4\}$

$\because$ $f(x) = x^2$

when $x = 2$, $f(2) = 2^2 = 4$

when $x = -2$, $f(-2) = (-2^2) = 4$

$\therefore$ $(2, -2)$ **Ans.**

(iii) $\{y \mid f(y) = -1\}$

$\because$ $f(x) = x^2$

$\Rightarrow$ $f(y) = y^2$

There is no value of $y$ for which $f(y)$ is $-1$

Hence result is $\phi$

■ **Example 7. Let $A = \{-2, -1, 0, 1, 2\}$ and the function $f : A \to R$ is defined by $f(x) = x^2 + 1$ find the range of $f$.**

**Sol.** Here $A = \{-2, -1, 0, 1, 2\}$ and $f(x) = x^2 + 1$

where $x = -2$ $f(-2) = (-2)^2 + 1 = 5$

$x = -1$ $f(-1) = (-1)^2 + 1 = 2$

when $x = 0$ $f(0) = 0^2 + 1 = 1$,

when $x = 1$ $f(1) = (1)^2 + 1 = 2$

when $x = 2$ $f(2) = 4 + 1 = 5$

Hence range of f are = $\{1, 2, 5\}$ **Ans.**

■ **Example 8. Let $A = \{-2, -1, 0, 1, 2\}$ and $f : A \to Z$, where $f(x) = x^2 - 2x - 3$ then find**

**(i) range of $f$** **(ii) the pre images of 6, – 3 and 5**

**Sol.** We are given that $A = \{-2, -1, 0, 1, 2\}$ and function

$f : A \to Z$ where $f(x) = x^2 - 2x - 3$

(i) $f(-2) = (-2)^2 - 2(-2) - 3 = 4 + 4 - 3 = 5$

$f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0$

$f(0) = (0)^2 - 2(0) - 3 = -3$,

$f(1) = (1)^2 - 2(1) - 3 = 1 - 2 - 3 = -4$

$f(2) = (2)^2 - 2(2) - 3 = 4 - 4 - 3 = -3$

range of $f = \{-4, -3, 0, 5\}$

(ii) From the above it is clear that 6 is not an image of any element. Therefore pre image of 6 will not exist.

Let per image of $-3$ be $x_1$ then $f(x) = -3$

$$= x^2 - 2x - 3 = -3 \quad = x^2 - 2x = 0$$

$$= x(x-2) = 0 \quad x = 0, 2$$

Similarly pre image of 5 is $-2$.

Hence pre image of 6, $-3$ and 5 are $\phi$, $\{0, -2\}$, $+2$

**□ Example 9. If $f : R \to R$ where $f(x) = e^x$ find :**

**(a) The $f$ – image of $R$**

**(b) $\{y \mid f(y) = 1\}$**

**(c) Is $f(x + y) = f(x) f(y)$ true ?**

**Sol.** (a) For all real values of $x$, $e^x$ is a positive real number. Also for any positive real value of $x$, there exist an element by $x$ in $R$ (domain) such that

$f(\log x) = e^{\log x} = x$

$\therefore$ $f$ image of $R$ or range of $f$ is $R^+$ **Ans.**

(b) $\quad f(y) = e^y = 1$

$\Rightarrow \quad e^y = e^o \Rightarrow y = 0$

$\therefore \quad \{y/f(y) = 1\} = \{0\}$

(c) $\because \quad f(x + y) = e^{x+y}, \ \forall\ x, y \in R$

$= e^x . e^y = f(x) . f(y)$

$\therefore \quad f(x + y) = f(x) . f(y)$ is true

**□ Example 10. If $f : R^+ \to R$ where $f(x) = \log x$, where $R^+$ is the set of positive real numbers, find**

**(a) $f(R^+)$** **(b) $\{y \mid f(y) = -2|\}$**

**(c) Is $f(x, y) = f(x) + f(y)$ true ?**

**Sol.**(i) $f : R^+ \to R$, and $f(x) = \log x$

Now for all $x \in R^+$, $\log x \in R$

$\therefore y = \log x \Rightarrow x = e^y \quad \therefore$ range of $f(x) = R$ **Ans.**

(ii) $\quad f(y) = -2$

$\Rightarrow \quad \log y = -2$

$\Rightarrow \quad y = e^{-2}$

$\therefore \quad \{y/f(y) = -2\} = \{e^{-2}\}$

(iii) Here $\quad f(x) = \log(x) = f(y) = \log y$

therefore $f(x. y) = \log(x y) = \log x + \log y$

$\Rightarrow \quad f(xy) = f(x) + f(y)$ is true.



**□ Example 11. If $f = \left\{\left(x, \frac{x^2}{1+x^2}\right), x \in R\right\}$ be a function from $R$ into $R$ ($R$ to $R$). Determine the range of f.**

**Sol.** Here $f$ is a function from $R$ to $R$ defined by

$$y = f(x) = \frac{x^2}{1+x^2} \qquad \text{....(i)}$$

Here y is defined $\forall$ real $x$

From (i) $y(1 + x^2) = x^2$ or $x^2(1 - y) = y$

$$\therefore \quad x^2 = \frac{y}{1-y} \quad \text{or } x = \pm\sqrt{\frac{y}{(1-y)}} \qquad \text{....(ii)}$$

Since $x$ is real $\Rightarrow \frac{y}{1-y} \geq 0 \Leftrightarrow 0 \leq y < 1$

Hence range = (0, 1)

**□ Example 12. If $g = \{(1,1), (2,3), (3,5), (4,7)\}$ a function ? If $g$ is represented formula $g(x) = \alpha x + \beta$, Find the value of $\alpha$ and $\beta$.**

**Sol.** According to question

$$g = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$$

Here g is function because the images of 1, 2, 3, 4 are respectively equal to 1, 3, 5, 7 which shows that every element has one image. Hence it is function.

Now $g(1) = 1, g(2) = 3, \ g(3) = 4, g(5) = 7$

from $g(x) = \alpha x + \beta$

$g(1) = \alpha. 1 + \beta = \alpha + \beta = 1 \qquad$ ....(i)

$g(2) = \alpha.2 + \beta = 2\alpha + \beta = 3 \qquad$ ....(ii)

Subtracting (i) from (ii) we get

$$2\alpha - \alpha = 3 - 1 \Rightarrow \alpha = 2$$

Now substituting the value of a = 2 in (i) we get

$$2 + \beta = 1 \Rightarrow \beta = -1$$

Hence $\quad \alpha = 2 \quad \beta = -1$ **Ans.**

**□ Example. 13. Classify the following function as one-one, many one into or onto give reasons.**

**(i) $f : Q \to Q, f(x) = 3x + 7$**

**(ii) $f : R \to [-1,1], f(x) = \sin x$ (iii) $f : N \to Z, f(x) = |x|$**

**Sol.** (i) Let $x_1, x_2 \in Q$ such that $x_1 \neq x_2$ then

$$x_1 \neq x_2$$

$\Rightarrow \quad 3x_1 \neq 3x_2$

$\Rightarrow \quad 3x_1 + 7 \neq 3x_2 + 7$

$\Rightarrow \quad f(x_1) = f(x_2)$

Hence $f$ is one-one

Again let $y \in Q$ (co-domain) if possible let its pre image under $f$ be $x$. Then $f(x) = y \Rightarrow 3x + y \Rightarrow x = \frac{y-7}{3}$. Now if $y \in Q$ then it is necessary that $\frac{y-7}{3} \in Q$

Hence $f$ is onto. Hence $f$ is one-one onto.

**(ii)** $f : R \to [-1, 1], f(x) = \sin x$

Let $x, y \in R$ then $f(x) = f(y)$

$\Rightarrow \quad \sin x = \sin y$

$\Rightarrow \quad x = n\pi + (-1)^n y$ w here $n \in z$

Hence, function is not one-one i.e. many one again, we know that $-1 \le \sin x \le 1$

$\therefore \quad f(R) = \{x \mid -1 \le x \le 1\}$

Hence $\quad f(R) \ne R$

Hence the function is into

Hence $f$ is many one into function

**(ii)** $f : N \to z, \quad f(x) = |x|$

We see that $x_1 = x + iy$ and $x_2 = x - iy$ $(y \ne 0)$ are different elements of the domain $C$.

$$f(x_1) = |x + iy| = \sqrt{x^2 + y^2}$$

$$f(x_2) = |x - iy| = \sqrt{x^2 + (-y^2)}$$

$$= \sqrt{x^2 + y^2}$$

$\Rightarrow \quad f(x_1) = f(x_2)$

Thus two different elements of the domain of $f$ have the same image hence $f$ is a many one function

Again range of $f = \{|x| : x \in N\} = z^+ \cup \{0\} \ne R$ (codomain)

$\therefore \quad f$ is not onto

Hence f is many one into function

**◘ Example 14. If $A = \{x \mid -1 \le x \le 1\} = B$ then tell which of the following functions. Defined from $A$ to $B$ are one-one, onto or one-one onto.**

**(i)** $f(x) = \frac{x}{2}$ **(ii)** $g(x) = |x|$

**(iii)** $h(x) = x^2$ **(iv)** $k(x) = \sin \pi x$

**Solution :** (i) If $x, y \in A$ then $f(x) = f(y)$

$\Rightarrow \quad \frac{x}{2} = \frac{y}{2} \Rightarrow x = y$

Hence $f$ is one-one

Again range of $f = f(A) = \left\{\frac{x}{2} \mid x \in A\right\}$



$$= \left\{|x| \frac{-1}{2} \le x \le \frac{1}{2}\right\} \ne B \text{ (Co domain)}$$

$\therefore \quad f$ is not onto

Therefore $f$ is one-one into function

**Solution :** (ii) $g(x) = |x|$

Function g is not injective (ie. one-one into) because the image of two different elements of $A$ may have the same $g$ image in $B$. For example $\frac{1}{2}, \frac{-1}{2} \in A$ are such two elements $\frac{1}{2} \ne \frac{-1}{2}$ but $g\left(\frac{1}{2}\right) = g\left(\frac{-1}{2}\right) = \frac{1}{2}$

Again $|x|$ is always non negative real number hence the range

$$g = g(A) = \{|x| \mid x \in A\}$$

$$= \{x \mid 0 \le x \le 1\} \ne B \text{ (Codomain)}$$

the function g is not surjective (or onto)

Hence, the function $g$ is not surjective (or onto)

Hence the function $g$ is injective but not surjective

(iii) Function h is not one-one because two different elements in $A$ may have equal h images.

For example : $\frac{1}{2}, \frac{-1}{2} \in A, \frac{1}{2} \ne \frac{-1}{2}$ but $h\left(\frac{1}{2}\right) = h\left(\frac{-1}{2}\right) = \frac{1}{4}$

Also $x^2$ will always be positive real number because x is a real number, consequently

$$h(A) = \{x^2 \mid x \in A\} = \{x \mid 0 \le x \le 1\} \ne B \text{ (Co-domain)}$$

$\therefore \quad$ function $h$ is not onto

Hence the function $h$ is neither one-one nor onto.

(iv) $k(x) = \sin \pi x$

Here $-1, 1 \in A$ are two numbers such that $-1 \ne 1$

But $k(-1) = \sin(-\pi) = -\sin \pi = 0$ and $k(1) = \sin \pi = 0$

$\therefore \quad -1, 1 \in A$ where $-1 \ne 1$ but $k(-1) = k(1)$

Hence function k is many one

But range of $k = k(A) = k(A) = \{\sin \pi x \mid x \in A\}$

$$= \{(x \mid -1 \le x \le 1)\} = B \text{ (Codomain)}$$

Hence $k$ is many one onto function.

**◘ Example 15. Give an example of each of the following types of functions**

(i) **One-one into function** (ii) **Many one onto function**
(iii) **Onto but not one-one** (iv) **One one but not onto**
(v) **Neither one-one nor onto** (vi) **One-one onto**

**Solutio : (i) One-one into function :** The functions $f: A \to B$ is said to be one-one into mapping if distinct elements in $A$ have distinct $f$ images in $B$ and there is at least one element in $B$. Which is not $f$ image of any element in the set $A$.

Hence $f: A \to B$ is one-one into iff $a \neq b \Rightarrow f(a) \neq f(b)$ $a, b \in A$ and $f(A) \neq B$

**Example :** If $A = \{3, 5, 7\}$ and $B = \{4, 9, 16, 25, 36, 49, 64\}$ and function $f: A \to B$ be defined such that $f(x) = x^2$ then under the function

$$f(3) = 3^2 = 9 \quad f(5) = 5^2 = 25$$

$f(7) = 7^2 = 49$ this is one-one into mapping since distinct element of $A$ have distinct image in $B$ and the elements 4, 16, 36, 64 are in $B$. Which are not an image of any element in $A$.

**(ii) Many one onto function :** The mapping $f: A \to B$ is said to be 'many one onto' if $f(a_1) = f(a_2)$ even if $a_1 \neq a_2$ $a_1, a_2 \in A$ and there is no element in $B$ which is not an image of an element in $A$.

**Example :** If $A = \{-2, 2, -1, 1\}$ and $B = \{4, 7\}$ then the function $f: A \to B$ is defined by $f(x) = x^2 + 3$ is many one onto, see now.

Here
$$f(-2) = (-2)^2 + 3 = 7, f(-2) = 2^2 + 3 = 7$$
$$f(-1) = (-1)^2 + 3 = 4, f(+1) = 1^2 + 3 = 4$$

Therefore this mapping is many one onto because here the image of the element 2 and $-2$ of $A$ is 7 in $B$ and the image of the elements $-1$ and 1 of $A$ is 4 in $B$ and there is no element in $B$ which is not an image of an element of $A$.

**(iii) Onto but not one-one :** For onto function the co-domain and range are same

Let $A = \{-1, 1, 3\}$, $B = \{2, 1, 0\}$ and let $f: A \to B$ be defined by $f(x) = x^2 + 1$ then $f(-1) = (-1)^2 + 1 = 2, f(1) = 1^2 + 1 = 2, f(3) = 3^2 + 1 = 10$

$\because$ $f(A) = B$

$\therefore$ The function is onto but for the function to be one-one, no two element of $A$ have the same image in $B$. But $-1$ and 1 of $A$ has the same image 2 in $B$. Hence function is not one-one hence function is onto but not one-one.

**(iv) One-one but not onto :**

$$f: z \to z, f(x) = 2x + 7$$

Let $x, y \in z$ then $f(x) = f(y)$

$\Rightarrow 2x + 7 = 2y + 7 \Rightarrow 2x = 2y \Rightarrow x = y$

Hence $f$ is one-one.

Again let $y \in z$ if possible, let $x$ be the pre image of $y$ then $f(x) = y$

$$\Rightarrow \quad 2x + 7 = y \text{ or } x = \frac{y-7}{2}$$



If y is an integer then $\frac{y-7}{2}$ is not necessarily an integer.

Hence the pre image of every element of z (codomain) does not exist in $z$ hence $f$ is not onto.

Hence $f$ is one-one but not onto.

**(v) Neither one-one nor onto :** Let $f: R \to R$ be the function defined by $f(x) = \sin x \ \forall \ x \in R$. Since for different values of $x \in R$, we can have the same value of $f(x)$ i.e. of $\sin x$ and for $\forall \ x \in R, f(x) = \sin x$ has the values in the closed interval $[-1, 1]$ so, $f$ is neither one-one nor onto.

**(vi) One-one onto or bijective :** Let $A = [1, 2]$ and $B = [1, 4]$ and $f: A \to B$ is defined by $f(x) = x^2 \ \forall \ x \in A$ then $f$ is one-one and onto therefore $f$ is bijective.

■ **Example 16. Prove that the function $f: R \to R$ defined by $f(x) = \cos x$ is many one into function. Modify the domain and codomain of $f$ such that f becomes.**

**(i) One-one into** **(ii) Many one onto**
**(iii) One-one onto**

**Sol.** Here we are given that $f: R \to R$, $f(x) = \cos x$

Since $-1 \leq \cos x \leq 1 \ \forall \ x \in R$ and $\cos x$ takes every value between $-1$ and 1 for some $x \in R$.

Hence we conclude that the image set of $R$ under $f = f(R) = \{x \in R | -1 \leq x \leq 1\} = \{-1, 1\}$

Again the image set of $R$ under $f$ is not equal to codomain $R$ of $f$. Hence the function is not onto ie. into.

Let $$f(x) = \frac{1}{2} \Rightarrow \cos x = \frac{1}{2} = \cos\frac{\pi}{3}$$

$$\Rightarrow \quad x = 2n\pi \pm \frac{\pi}{3}, \ n \in I$$

$$\therefore \quad \{x/x \in R \ f(x) = \frac{1}{2}\} = \{2n\pi \pm \frac{\pi}{3}, n \in I\}$$

Here we see that $\frac{1}{2}$ is the $f$ image of many elements of $R$. Hence $f$ is many one into function.

Now we have to change domain of $f$ in such way that $f$ become

**(i) One-one into :** If function be defined in $f: [0, \pi] \to R$ then for any two values of domain $[0, \pi]$ there is no image in $R$. Therefore

$$f(x_1) = f(x_2), \ x_1, x_2 \in [0, \pi]$$

$$\cos x_1 = \cos x_2 \quad \Rightarrow \quad x_1 = x_2$$

$\Rightarrow$ $f$ is one-one and for its codomain $R$. We have already proved that it is into.

**(ii) Many one onto :** If function be defined in $f: R \to [-1, 1]$ then since $-1 \leq \cos x \leq 1$

Again the image set of $R$ under f is $[-1, 1]$ which is equal to its codomain. Hence $f$ is onto and for the domain $R$ we have already proved that it is many one.

**(iii) One-one onto :** If function be defined in

$f: [0, \pi] \to R$ then $f(x_1) = f(x_2)$ $x_1, x_2 \in [0, \pi]$

$\Rightarrow \cos x_1 = \cos x_2 \Rightarrow x_1 = x_2$

Hence $f$ is one-one but we know that $-1 \leq \cos x \leq 1$ again the image set of $R$ under $f$ is $[-1, 1]$ which is equal to its codomain. Hence $f$ is onto. Hence $f$ is one-one onto

**◘ Example 17. If $N = \{1,2,3,4,.....\}$, $O = \{1,3,5,7,.....\}$ $E = \{2,4,6,8,.....\}$ and $f_1$, $f_2$ are functions**

**$f_1 : N \to O, \quad f_1(x) = 2x - 1$**

**$f_2 : N \to E, \quad f_2(x) = 2x$**

**prove that $f_1$ and $f_2$ are one-one onto.**

**Solution : Case I :** Here $f_1 : N \to O, f_1(x) = 2x - 1$ where $N = \{1, 2, 3, 4 ......\}$ and $O = \{1, 3, 5, 7, ......\}$

**One-one function :** Let $f(x_1) = f(x_2)$, $x_1, x_2 \in N$

$\Rightarrow 2x_1 - 1 = 2x_2 - 1 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ $f$ is one-one

**Onto function :** Let $y \in O$ be an element in set O whose pre image $x \in N$.

then $f_1(x) = 2x - 1 = y \Rightarrow x = \dfrac{y+1}{2}$

Here we see that for every value of $y$ (in O) $\exists$ a value $x$ in $N$ ie. the pre image of each element of O (Codomain) exists in the codomain of $N$.

Hence $f$ is onto

**Case II** Here $f_2 : N \to E, f_2(x) = 2x$

where $N = \{1, 2, 3, 4\}$ and $E = \{2, 4, 6, 8\}$

For one-one function let $x_1, x_2 \in N$

Let $x_1 = x_2$

$\Rightarrow 2x_1 = 2x_2 \Rightarrow f_1(x) = f_2(x)$

$\therefore$ $f$ is one-one

For onto function

Let $y \in E$ be an element, if possible, let the pre-image of $y$ be $x$ in $N$.

then $f_2(x) = 2x = y \Rightarrow x = y/2$

Here we see that for every value of $y$ in E $\exists$ a value $x$ in $N$. i.e. the pre image of every element of the codomain E exist in the domain $N$. Hence $f_2$ is onto.

Thus $f_2 : N \to E$ is one one onto function.



**◘ Example 18.If a function f is defined from the set $R$ of real numbers to $R$ as follows classify $f$ as one-one many into or onto give reason.**

**(i) $f(x) = x^2$ (ii) $f(x) = x^3$ (iii) $f(x) = x^3 + 3$ (iv) $f(x) = x^3 - x$**

**Sol.** (i) Since $f: R \to R$, is defined $f(x) = x^2$

then $f^{-1}(4) = \{2, -2\}$

because 4 is the image of both 2 and $-2$

Hence the function $f$ is not one-one it is many one since there is not element in $R$. Whose square is $-4$ therefore $f^{-1}(-4) = \phi$. The range of $f$ is the set of all non negative real number. Therefore we cannot find elements in $R$. Whose square is negative real number. Hence $f$ is not onto.

Hence $f: R \to R$ is many one into function

(ii) $f: R \to R, f(x) = x^3$

If $x_1, x_2 \in R$ such that $x_1 \neq x_2$ then $x_1 \neq x_2$

$\Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$

$\therefore$ The function $f$ is one-one

Let $y \in R$ then $y^{1/3} \in R$ and we have

$f(y)^{1/3} = (y^{1/3})^3 = y$

$\therefore$ $f$ is onto.

Hence $f$ is one-one and onto ie bijective.

(iii) Let $x, y \in R$ then

$f(x) = f(y) = x^3 + 3 = y^3 + 3 \Rightarrow x^3 = y^3 \Rightarrow x = y$

Hence $f$ is one one

Again let $y \in R$ (co domain)

If possible let $x$ be the pre image of y. Then

$f(x) = y = x^3 + 3 = y \Rightarrow x = (y-3)^{1/3} \in R$

Hence the pre image of every element of $R$ (Co-domain) exists in $R$ (domain) hence $f$ is onto.

(iv) Let $x, y \in R$ then

$f(x) = f(y) \Rightarrow x^3 - x \Rightarrow y^3 - y$

$\Rightarrow (x^3 - y^3) - (x - y) = 0$

$\Rightarrow (x - y)[x^2 + xy + y^2 - 1] = 0$

either $x = y$ or $x^2 + xy + y^2 - 1 = 0$

Hence function $f$ is many one

Again we see that $y \in R$ (codomain). If possible let $x$ be the pre image of $y$ under $f$. Then $f(x) = y$

$\Rightarrow x^3 - x = y \Rightarrow x^3 - x - y = 0$

This equation has at least one real root therefore pre image of each element of codomain $R$ exists in the domain $R$. Hence function is onto.

Hence $f$ is many one onto.

**□ Example 19.** If $A = \{1,2,3,4\}$ and $f = \{(1,2), (2,1), (3,3), (4,2),\}$, $g = \{(1,3), (2,1), (3,2), (4,4)\}$ are two functions defined on $A$. Find

(i) $fog$ (ii) $gof$ (iii) $fof$

(i) $fog\ (1) = f\ [g\ (1)] = f\ (3)$

$\therefore$ $fog\ (1) = 3$

$fog\ (2) = f\ [g\ (2)] = f\ (1)$

$\therefore$ $fog\ (2) = 2$

$fog\ (3) = f\ [g\ (3)] = f\ (2)$

$\therefore$ $fog\ (3) = 1$

and $fog\ (4) = f\ [g\ (4)] = f\ (4)$

$\therefore$ $fog\ (4) = 2$

$\therefore$ $fog = \{(1, 3), (2, 2), (3, 1), (4, 2)\}$ **Ans.**

(ii) $gof\ (1) = g\ [f(1)] = g\ (2)$

$\therefore$ $gof\ (1) = 1$

$gof\ (2) = g\ [f(2)] = g\ (1)$

$\therefore$ $gof\ (2) = 3$

$gof\ (3) = g\ [f(3)] = g\ (3)$

$\therefore$ $gof\ (3) = 2$

and $gof\ (4) = g\ [f(4)] = g\ (2)$

$\therefore$ $gof\ (4) = 1$

$\therefore$ $gof = \{(1, 1), (2, 3), (3, 2), (4, 1)\}$ **Ans.**

(iii) $fof\ (1) = f\ [f(1)] = f\ (2)$

$\therefore$ $fof\ (1) = 1$

$fof\ (2) = f\ [f(2)] = f\ (1)$

$\therefore$ $fof\ (2) = 2$

$fof\ (3) = f\ [f(3)] = f\ (3)$

$\therefore$ $fof\ (3) = 3$

$fof\ (4) = f\ [f(4)] = f\ (2)$

$\therefore$ $fof\ (4) = 1$

$\therefore$ $fof = \{(1, 1), (2, 2), (3, 3), (4, 1)\}$ **Ans.**

**□ Example 20. If $f : R \to R$ and $g : R \to R$ are two functions defined as follows then find $(fog)(x)$ and $(gof)(x)$ :**

**(i) $f(x) = 2x + 3$, $g(x) = x^2 + 1$** (ii) $f(x) = x^2 + 8$, **$g(x) = 3x^3 + 1$**

**(iii) $f(x) = \frac{1}{2-x}$, $g(x) = \frac{1}{1+x}$** (iv) $f(x) = x^2 + 3x + 1$, **$g(x) = (2x - 3)$**

**(v) $f(x) = x(x-1)$, $g(x) = x(x+1)$**

**(vi) $f(x) = x + 7$, $g(x) = \frac{1}{x^2+1}$**



**Sol.** (i) From the definition of the functions $f$ and $g$ it is obvious that $gof : R \to R$ can be defined

(i) $(gof)(x) = g[f(x)]$

$= g(2x + 3) = (2x + 3)^2 + 1$

$= 4x^2 + 12x + 9 + 1$

$= 4x^2 + 12x + 10$

$(fog)(x) = f[g(x)]$

$= f[x^2 + 1] = 2(x^2 + 1) + 3 = 2x^2 + 2 + 3 = 2x^2 + 5$ **Ans.**

(ii) $f(x) = x^2 + 8$, $g(x) = 3x^3 + 1$

$fog\ (x) = f\ [g\ (x)]$

$= f\ [3x^3 + 1]$

$= (3x^3 + 1)^2 + 8$

$= 9x^6 + 6x^3 + 1 + 8 = 9x^6 + 6x^3 + 9$

$gof\ (x) = g\ [f(x)]$

$= g\ [x^2 + 8]$

$= 3\ (x^2 + 8)^3 + 1$

$= 3\ [x^6 + 24x^4 + 192x^2 + 512] + 1$

$= 3x^6 + 72x^4 + 576x^2 + 1536 + 1$

$= 3x^6 + 72x^4 + 576x^2 + 1537$ **Ans.**

(iii) $f(x) = \frac{1}{2-x}$, $g(x) = \frac{1}{1+x}$

$fog\ (x) = f\ [g\ (x)]$

$= f\left[\frac{1}{x+1}\right]$

$= \frac{1}{2-\frac{1}{x+1}} = \frac{1+x}{2(1+x)-1}$

$\therefore$ $(fog)(x) = \frac{x+1}{2x+2-1} = \frac{1+x}{2x+1}$

$gof\ (x) = g\ [f(x)]$

$= g\left[\frac{1}{2-x}\right]$

$= \frac{1}{1+\frac{1}{(2-x)}} = \frac{2-x}{2-x+1}$ **Ans.**

(iv) $f(x) = x^2 + 3x + 1$, $g(x) = (2x - 3)$

$fog\ (x) = f\ [g\ (x)]$

$$= f[(2x-3)] = (2x-3)^2 + 3(2x-3) + 1$$
$$= 4x^2 - 12x + 9 + 6x - 9 + 1$$
$$= 4x^2 - 6x + 1$$

$$gof(x) = g[f(x)]$$
$$= g[x^2 + 3x + 1] = 2(x^2 + 3x + 1) - 3$$
$$= 2x^2 + 6x + 2 - 3$$
$$= 2x^2 + 6x - 1 \quad \textbf{Ans.}$$

(v) $f(x) = x(x-1), \quad g(x) = x(x+1)$

$$fog(x) = f[g(x)] = f[x(x+1)]$$
$$= [x(x+1)]\,[x(x+1) - 1]$$
$$= [x^2 + x]\,[(x^2 + x) - 1] = (x^2 + x)^2 - (x^2 + x)$$
$$= x^4 + x^2 + 2x^3 - x^2 - x \quad \text{or} \quad (x^2 + x)(x^2 + x - 1) \quad \textbf{Ans.}$$

$$gof(x) = g[f(x)]$$
$$= g[x(x-1)] = \{x(x-1)\}\,[x(x-1) + 1]$$
$$= (x^2 - x)^2\,[(x^2 - x) + 1] = (x^2 - x)^2 + (x^2 - x)$$
$$= x^4 + x^2 - 2x^3 + x^2 - x$$

or
$$= (x^2 - x)(x^2 - x + 1) \quad \textbf{Ans.}$$

(vi) $f(x) = x + 7, \; g(x) = \dfrac{1}{x^2+1}$

$$fog(x) = f[g(x)] = f\left[\frac{1}{x^2+1}\right]$$
$$= \frac{1}{x^2+1} + 7 = \frac{1 + 7x^2 + 7}{x^2+1}$$
$$= \frac{7x^2+8}{(x^2+1)}$$

and
$$gof(x) = g[f(x)]$$
$$= g[x+7]$$
$$= \frac{1}{(x+7)^2+1}$$
$$= \frac{1}{x^2+14x+49+1} = \frac{1}{x^2+14x+50} \quad \text{Ans.}$$

**◘ Example 21. If $f : R^+ \to R^+$, $f(x) = x^2 + \frac{1}{x^2}$ $g : R^+ \to R^+$, $g(x) = e^x$ find $(gof)(x)$**

Sol. Here we have

$$f : R^+ \to R^+, f(x) = x^2 + \frac{1}{x^2}$$



and $g : R^+ \to R^+, g(x) = e^x$

$$\therefore \quad (gof)(x) = g[f(x)]$$
$$= g\left[x^2 + \frac{1}{x^2}\right]$$
$$= e^{x^2 + \frac{1}{x^2}} = e^{x^2 + x^{-2}} = e^{x^2} . e^{x^{-2}} \quad \text{Ans.}$$

**◘ Example 22. If $f(x) = \log \dfrac{1+x}{1-x}$ and $g(x) = \dfrac{3x+x^3}{1+3x^2}$ find $(fog)(x)$.**

Sol. Here we have $f(x) = \log \dfrac{1+x}{1-x}$, $g(x) = \dfrac{3x+x^3}{1+3x^2}$

$$\therefore \quad (fog)(x) = f[g(x)] = f\left[\frac{3x+x^3}{1+3x^2}\right]$$

$$fog(x) = \log\left\{\frac{1 + \frac{3x+x^3}{1+3x^2}}{1 - \frac{3x+x^3}{1+3x^2}}\right\}$$

$$= \log\left\{\frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}\right\}$$

$$\log\frac{(x+1)^3}{(1-x)^3} = \log\left(\frac{x+1}{1-x}\right)^3$$
$$= 3\log\left(\frac{1+x}{1-x}\right)$$

**◘ Example 23. If $f : R^+ \to R^+$, $f(x) = x^2$ and $g : R^+ \to R^+$, $g(x) = \sqrt{x}$ then find $gof$ and $fog$ are they equal functions.**

Sol. Here we have $f : R^+ \to R^+$, $g(x) = \sqrt{x}$

$$\therefore \quad gof(x) = g[f(x)]$$
$$= g(x^2) = \sqrt{x^2} = x$$

and
$$fog(x) = f[g(x)]$$
$$= f(\sqrt{x}) = (\sqrt{x})^2 = x$$

$$\therefore \quad gof(x) = fog(x) = (x)$$

Hence they are equal functions.

**◘ Example 24. If $f : A \to B$ and $g : B \to C$** be two such functions that $(gof) : A \to C$ **is defined, then proved that**

**(i) If $gof$ is one-one then $g$ is also one-one**
**(ii) If $gof$ is one one then $f$ is also one-one**
**(iii) If $gof$ is onto and $g$ is one-one then $f$ is onto**

**Sol. (i)** $f : A \to B$ and $g : B \to C$ then the function $gof$ can be defined from set $A$ to set $C$ i.e. $(gof)\ A \to C$

(i) If $gof$ is bijection then $g$ is also bijection, $gof$ is bijection means, every element of $C$ is $gof$ image of some element of $A$.

i.e. let $x \in A$ and $y \in C$ then for each element $x$ there exist at least one element in $y \in C$ such that

$$(gof)\ (x) = y$$

$\Rightarrow$ $g\ [f\ (x)] = y$

i.e. for each $y$ there correspond $f(x)$ present in $B$ hence pre image of every element of $C$ is in $B$.

Hence function $g : B \to C$ is onto

(ii) If $gof$ is one-one then $f$ is also one-one

Let $x_1, x_2$ are two elements in $A$ be such that

$$gof\ (x) = (gof)\ (x_2)$$

then $gof(x_1) = gof(x_2) \Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow f(x_1) = f(x_2)$ [$\because$ $g$ is one one]

$\Rightarrow x_1 = x_2$ [$\because$ $f$ is one one]

Hence function $f$ is one-one. If $gof$ is one-one.

(iii) If $gof$ is onto and $g$ is one-one then $f$ is onto

If $gof$ is onto i.e. pre image of every element of $C$ lie in $A$ and $g$ is one-one i.e.

If $g\ (x_1) = g\ (x_2)$, $x_1, x_2, \in B$

$\Rightarrow$ $x_1 = x_2$

and $gof\ (x) = y$, $y \in C,\ x \in A$

$\Rightarrow$ $g\ [f\ (x)] = y$

Let $g\ \{f\ (x_1)\} = y_1$ and $g\ \{f\ (x_2)\} = y_2$,

According to question

$$y_1 = y_2$$
$$f\ (x_1) = f\ (x_2)$$

Hence a pre-image exists for each corresponds to $y$ and there pre image are also same for any two equal values of $y$. Hence for pre-image of every element of $B$ will in $A$. Hence $f$ is onto.

■ **Example 25. If $f : R \to R, f\ (x) = 2x - 3$ and $g : R \to R$ $g(x) = \dfrac{x+3}{2}$, prove that $fog = gof = I_R$.**

**Sol.** Hence $f : R \to R$, $f(x) = 2x - 3$



and $g : R \to R$, $g(x) = \dfrac{x+3}{2}$

$\therefore$ $(fog)\ (x) = f\ [g\ (x)]$

$$= f\left(\frac{x+3}{2}\right) = 2.\ \left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$$

Also $(gof)\ (x) = g\ [f\ (x)]$

$$= g\ [2x - 3]$$
$$= \frac{2x-3+3}{2} = x$$
$$I_R = I_R\ (x) = x$$

$\therefore$ $gof\ (x) = fog\ (x) = I_R\ (x) = x$

Hence $gof = fog = I_R$ **Hence proved**

■ **Example 26. For which of the following functions, the inverse function exists ? Give reasons :**

**(i) $f : R \to R, f\ (x) = e^x$** **(ii) $g : R \to R, g\ (x) = |\ x\ |$**

**(iii) $h : [0, \pi] \to [-1, 1], h\ (x) = \cos x$** **(iv) $\phi : Q \to Q, \phi\ (x) = x^3 - 1$**

**Sol. (i)** $f : R \to R, f(x) = e^x$

Here $f(x) = e^x = y$

$\Rightarrow$ $x = \log y$

Correspond to every value of $x$, $y$ will be positive. Therefore image set of $R$ under $f = f(R)$

$$= \{x \in R^+\}$$

Again image set of $R$ under $f$ is not equivalent to codomain $R$ of $f$. Hence $f$ is not onto.

Hence inverse function of $f$ doesn't exist.

(ii) $g : R \to R, g(x) = |\ x\ |$

Correspond to every value of $x$, $y$ will be positive therefore image set of $R$ under $f = f(R)$

$$f = R^+$$

Again image set of $R$ under $f$ is not equivalent to codomain $R$ of $f$. Hence $f$ is not onto.

Hence inverse function of $f$ does not exist.

(iii) $h : [0, \pi] \to [-1, 1], h(x) = \cos x$

Here $h(x) = \cos x$

Let $x_1\ x_2 \in [0, \pi]$

If $h(x_1) = h(x_2)$

$\Rightarrow \cos x_1 = \cos x_2 \Rightarrow x_1 = x_2$

Hence $h$ is one-one

We know that $-1 \le \cos x \le 1$

Again image set $[-1, 1]$ of $x$ under $h$ is equivalent to codomain of $h$. Hence $h$ is onto.

Hence inverse function of $h$ exist.

(iv) $\phi : Q \to Q, \phi(x) = x^3 - 1$

Let $x_1\ x_2 \in Q$

$$\phi(x_1) = \phi(x_2)$$

$\Rightarrow$ $x_1^3 - 1 = x_2^3 - 1 \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2$

Hence $\phi$ is one one

Again let $\phi(x) = y$

$\therefore$ $\phi(x) = x^3 - 1 = y$

$\Rightarrow$ $x^3 = y + 1$

$\Rightarrow$ $x = (y+1)^{1/3}$

i.e. correspond to every of $y$ the value of $x$ lie in $Q$ is not necessary.

For example there is no such value of $x$ in $Q$ for which $x^3 - 1 = 1$, Hence $\phi$ is not onto

Hence inverse function $\phi$ does not exist.

**■ Example 27. Find $f^{-1}$ (if it exist) when $f : A \to B$, where**

**(i) $A = \{0, -1, -3, 2\}$, $B = (-9, -3, 0, 6)$, $f(x) = 3x$**

**(ii) $A = \{1, 3, 5, 7, 9\}$, $B = \{0, 1, 9, 25, 49, 81\}$, $f(x) = x^2$**

**(iii) $A = \left\{x \in R \middle| -\frac{\pi}{2} \le x \le \frac{\pi}{2}\right\}$, $B = \{x \in R \mid -1 \le x \le 1\}$, $f(x) = \sin x$**

**(iv) $A = B = R, f(x) = x^3$**

**Sol.** (i) Here we have $f : A \to B$

$\therefore$ $f^{-1} : B \to A$

$$A = \{0, -1, -3, 2\} \text{ and } B = \{-9, -3, 0, 6\},$$

$f(x) = 3x$

$\therefore$ $f(0) = 0$

$f(-1) = -3$

$f(-3) = -9$

$f(2) = 6$

Here for all $x \in A$ $f(x) \in B$

$\therefore$ $f = \{(0, 0), (-1, -3), (-3, -9), (2, 6)\}$

$\therefore$ $f^{-1} = \{(0, 0), (-3, -1), (-9, -3), (6, 2)\}$

(ii) Here $A = \{1, 3, 5, 7, 9\}, B = \{0, 1, 9, 25, 49, 81\}$

Let $f(x) = x^2 = y$



$\Rightarrow$ $x^2 = f^{-1}(y)$ and $x = \pm\sqrt{y}$

$\Rightarrow$ $f^{-1}(y) = \pm\sqrt{y} \Rightarrow f^{-1}(0) = 0 \notin A$

i.e. $0 = y \in B$ But $f^{-1}(y) = f^{-1}(0) = 0 \in A$

Hence $f^{-1}$ does not exist.

(iii) $A = \left\{x \in R \middle| \frac{-\pi}{2} \le x \le \frac{\pi}{2}\right\}$, $B = \{x \in R \mid -1 \le x \le 1\}$

and $f(x) = \sin x$

$\because f : A \to B$ and $f(x) = \sin x$

Let $f^{-1}(x) = y$

$f(y) = x$

$\Rightarrow$ $\sin y = x$

$\Rightarrow$ $y = \sin^{-1} x$

$\therefore$ $f^{-1}(x) = \sin^{-1} x$

(iv) $A = B = R, f(x) = x^3$

let $y = f(x) \Rightarrow x = f^{-1}(y)$ and $y = x^3$

$\Rightarrow$ $x = y^{1/3} \Rightarrow f^{-1}(y) = y^{1/3}$

$\Rightarrow$ $f^{-1}(x) = x^{1/3}$

and $f$ is one-one onto. Therefore $f^{-1}$ exist and $f^{-1}(x) = x^{1/3}$.

**■ Example 28. If $f : R - \{-1\} \to R - \{-1\}, f(x) = \frac{x}{x+1}$, prove that $f$ is a bijection. Also find $f^{-1}$.**

**Sol.** If $x, y \in R - \{-1\}$ then $f(x) = f(y)$

$$\frac{x}{x+1} = \frac{y}{y+1}$$

$\Rightarrow$ $x(y+1) = y(x+1) \Rightarrow xy + x = yx + y \Rightarrow x = y$

Hence $f$ is one one whose pre image in $x \in R - \{-1\}$

Again let there exist an element $y \in R - \{-1\}$

then $f(x) = y$

$\Rightarrow$ $\frac{x}{x+1} = y \Rightarrow x = xy + y$

$\Rightarrow$ $x(1-y) = y \Rightarrow x = \frac{y}{1-y}$

For each value of $y$ then correspond a value $x$ except $y = 1$ i.e.

Pre image of every element of the codomain $R - \{-1\}$ exist in the domain $R - \{-1\}$

$\therefore$ $f$ is onto

Hence function $f$ is one-one onto

If $f$ image of $x$ is $y$ then

$f^{-1} : R-\{-1\} \Rightarrow R-\{-1\}$ will be defined as follows

$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$

$\Rightarrow \quad \dfrac{x}{x+1} = y \quad \Rightarrow \quad x = \dfrac{y}{1-y}$

Again $\quad f^{-1}(y) = x \quad \Rightarrow \quad f^{-1}(y) = \dfrac{y}{1-y}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{x}{1-x}$ **Ans.**

■ **Example 29. If $f(x) = \dfrac{x-1}{x+1}$, $x \neq -1, 1$ then prove that $f \, o \, f^{-1}$ is the identity function.**

**Sol.** Here we have $f(x) = \dfrac{x-1}{x+1}$, $x \neq -1, 1$

To find $\quad fof^{-1} = ?$

Here the given function $f$ is bijective so its inverse exist

Let $\quad f(x) = \dfrac{x-1}{x+1} = y$

$\Rightarrow \quad x - 1 = y(x+1) \Rightarrow x - 1 = yx + y$

$\Rightarrow \quad x - xy = 1 + y \Rightarrow x(1-y) = 1 + y$

$\Rightarrow \quad x = \dfrac{1+y}{1-y}$

If $f$ image of $y$ is $x$ then $f^{-1}$ will be defined as follows.

$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$

$\dfrac{x-1}{x+1} = y \quad \Rightarrow \quad x = \dfrac{1+y}{1-y}$

Again $\quad f^{-1}(y) = x \quad \Rightarrow \quad f^{-1}(y) = \dfrac{1+y}{1-y}$

$\Rightarrow \quad f^{-1}(x) = \dfrac{1+x}{1-x} \quad \forall x \neq -1, 1$

Now $\quad fof^{-1}(x) = f[f^{-1}(x)] = \left[\dfrac{1+x}{1-x}\right]$

$= \dfrac{\frac{1+x}{1-x} - 1}{\frac{1+x}{1-x} + 1}$

$= \dfrac{1+x-1+x}{1+x+1-x} = \dfrac{2x}{2} = x$



Therefore $fof^{-1}(x) = x = I(x)$

Hence $fof^{-1}$ is the identity function.

■ **Example 30. If $A = \{1,2,3,4\}$, $B = \{3,5,7,9\}$, $C = \{7,23,47,79\}$ and $f : A \to B$, $f(x) = 2x+1$; $g : B \to C$, $g(x) = x^2-2$ then write $(g \, of)^{-1}$ and $f^{-1} og^{-1}$ as set of ordered pairs.**

**Sol.** Here we are given that

$A = \{1, 2, 3, 4\}, B = \{3, 5, 7, 9\}, C = \{7, 23, 47, 79\}$

For $f$:

$f : A \to B, \quad f(x) = 2x + 1, \ \forall x \in A$

Now substituting the value of $x = 1, 2, 3, 4$ in $f(x) = 2x + 1$

when $x = 1 \quad f(x) = f(1) \quad 3 \in B$

when $x = 2 \quad f(2) = 2\times2 + 1 = 5 \in B$

when $x = 3 \quad f(3) = 2\times3 + 1 = 7 \in B$

when $x = 4 \quad f(4) = 2\times4 + 1 = 9 \in B$

$\therefore \quad f = \{(1, 3), (2, 5), (3, 7), (4, 9)\}$ ....(i)

Now range of $f$ = set of second components = $\{3, 5, 7, 9\} = B$

$\therefore$ $f$ is onto. Again first element of no two ordered pairs are same. Therefore $f$ is one-one. This way : $f : A \to B$ is one-one onto function therefore inverse function $f^{-1} : B \to A$ will exist then

$f^{-1} = \{(3,1)(5, 2), (7, 3)(9, 4)\}$ ....(ii)

will be one-one onto.

For $g$ :

Given that $g : B \to C$, $g(x) = x^2 - 2$ where $x \in B$

Now substituting the value of $x = 3, 5, 7, 9$ in $g(x)$

$g(x) = x^2 - 2$

when $x = 3$ then $g(3) = 3^2 - 2 = 7 \in C$

when $x = 5$ then $g(5) = 5^2 - 2 = 23 \in C$

when $x = 7$ then $g(7) = 7^2 - 2 = 47 \in C$

when $x = 9$ then $g(9) = 9^2 - 2 = 79 \in C$

$\therefore \quad g = \{(3, 7), (5, 23), (7, 47), (9, 79)\}$ ....(iii)

$\therefore$ range of $g$ = set of second elements = $\{7, 23, 47, 79\} = C$

$\therefore$ $g$ is onto since first element of no two ordered pairs are same.

Therefore $g$ is one-one. This way $g : B \to C$ is one-one onto. Therefore inverse function of $g$

i.e. $\quad g^{-1} : C \to B$ will exist

$\therefore \quad g^{-1} = \{(7, 3)(23, 5)(47, 7)(79, 9)\}$ ....(iv)

Will also be one-one onto function

Clearly the composite function of two bijections is also a bijection.

$\therefore$ $gof : A \to C$ is also a bijection and then its inverse function $(gof)^{-1} : \to A$ will exist.

Now $gof(x) = g[f(x)]$

$\therefore$ $(gof)(1) = g[f(1)] = g(3) = 7$

$(gof)(2) = g[f(2)] = g(5) = 23$

$(gof)(3) = g[f(2)] = g(7) = 47$

$(gof)(4) = g[f(4)] = g(9) = 79$

$\therefore$ $gof = \{(1, 7), (2, 23), (3, 47), (4,79)\}$

$gof^{-1} = \{(7, 1) (23, 2), (47, 3) (79, 4)\}$ .....(v)

Now if $g^{-1} B \to A$ and $f^{-1} B \to A$ then $f^{-1} og^{-1} : C \to A$ will exist and

$(f^{-1} og^{-1})(7) = f^{-1}\{g^{-1}(7)\} = f^{-1}(3) = 1$

$(f^{-1} og^{-1})(23) = f^{-1}\{g^{-1}(23)\} = f^{-1}(5) = 2$

$(f^{-1} og^{-1})(47) = f^{-1}\{g^{-1}(47)\} = f^{-1}(7) = 3$

$(f^{-1} og^{-1})(79) = f^{-1}\{g^{-1}(79)\} = f^{-1}(9) = 4$

$\therefore$ $(f^{-1} og^{-1}) = \{(7, 1), (23, 2), (47, 3) (79, 4)\}$ ....(vi)

Hence from (v) and (vi) we can also conclude that $(gof)^{-1} = f^{-1} og^{-1}$

■ **Example 31. If $f : R \to R, f(x) = 3x - 4$, does $f^{-1}$ exist ? If yes find the formula for $f^{-1}$.**

**Sol.** Let $x_1, x_2 \in R$ then $f(x_1) = f(x_2)$

$\Rightarrow$ $= 3x_1 - 4 = 3x_2 - 4$

$\Rightarrow$ $x_1 = x_2$

Hence $f$ is one one function

Again let $y \in R$ (codomain), if possible, let $x$ be the pre image of y then $f(x) = y$

$\Rightarrow$ $3x - 4 = y$ $\Rightarrow$ $x = \frac{y+4}{3} \in 4$ (domain)

Hence pre-image of every element of the co-domain $R$ exist in the domain $R$. Hence $f$ is onto. Thus $f$ is bijection hence the inverse $f^{-1} : R \to R$ exist let $y \in R$ and $f^{-1}(y) = x$ then $f(x) = y$

$\Rightarrow$ $3x - 4 = y$ $\Rightarrow$ $x = \frac{y+4}{3}$

$\Rightarrow$ $f^{-1}(y) = \frac{y+4}{3}$

$\Rightarrow$ $f^{-1}(x) = \frac{x+4}{3}$ $\forall\, x \in R$

Hence $f : R \to R, f^{-1}(x) = \frac{x+4}{3}$ Ans.

■ **Example 32.** Let $A = R - \{3\}$, $B = R - \{1\}$ and $f : A \to B$, $f(x) = \frac{x-2}{x-3}$ **prove that $f$ is is bijection. Also find the formula for $f^{-1}$.**



**Sol.** If $x_1, x_2 \in A$ be such that $f(x_1) = f(x_2)$

$\Rightarrow$ $\frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$

$\Rightarrow$ $(x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$

$\Rightarrow$ $x_1x_2 - 2x_2 - 3x_1 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$

$-3x_1 - 2x_2 = -3x_2 - 2x_1$

$\Rightarrow$ $-x_1 = -x_2$

$\Rightarrow$ $x_1 = x_2$

Hence $f$ is one-one

Let $y \in B$, if $x$ be the pre-image of $y$ then

$f(x) = \frac{x-2}{x-3} = y$

$\Rightarrow$ $x - 2 = y(x - 3)$

$\Rightarrow$ $x - 2 = xy - 3y$

$\Rightarrow$ $x = \frac{2-3y}{1-y}$ or $\frac{3y-2}{y-1}$

Here we see that the pre image of every element of the co-domain $B = R - \{1\}$ exist in the domain

$A = R - \{3\}$ Hence $f$ is onto

$\Rightarrow$ $f$ is one-one onto

So, their inverse exist.

If $f$ image of $x$ is $y$ then $f^{-1} : R \to R$ will defined as follows.

$f^{-1}(y) = x \Leftrightarrow f(x) = y$

$f(x) = \frac{x-2}{x-3}$

$\Rightarrow$ $y = \frac{x-2}{x-3}$ $\Rightarrow$ $x = \frac{3y-2}{y-1}$

Again $f^{-1}(y) = x$ $\Rightarrow$ $f^{-1}(y) = \frac{3y-2}{y-1}$

$\Rightarrow$ $f^{-1}(x) = \frac{3x-2}{x-1}$

$\Rightarrow$ $f^{-1} : B \to A$, $f^{-1}(x) = \frac{3x-2}{x-1}$ Ans.

■ **Example 33. Which of the following definitions of * is a binary operation on the set gives against. Give reason in support of your answer.**

**(iii) Find the invertible elements of $R$ with respect to $*$.**

**Sol.** (i) if $a, b \in R$ they by definition of $*$

$$a * b = a + b - ab = b + a - b.a = b * a$$

[by commutative property of addition and multiplication of real numbers.]

$\therefore$ $*$ is commutative.

Again $\quad a * (b * c) = a * (b + c - bc) = a + (b + c - bc) + a(b + c - bc)$

$$= a + b + c - ab - bc - ca + abc \quad \text{....(i)}$$

and $\quad (a * b) * c = (a + b - c) * c = a + b - ab + c - (a + b - ab)c$

$$= a + b + c - ab - bc - ca + abc \quad \text{....(ii)}$$

from (i) and (ii)

$$a * (b * c) = (a * b) * c, \ \forall a, b, c \in R$$

Hence $*$ is an associative operation.

(ii) If possible, let $c$ be the identity in $R$ for the operation $*$.

Thus, for any $a \in R$,

$$a * e = a \text{ [by definition of identity]}$$

$\Rightarrow \quad a + e - ae = a \quad \Rightarrow \quad e(1 - a) = 0$

$\Rightarrow \quad e = 0$ [$\because$ $(1 - a)$ is not necessarily equal to zero]

$\therefore$ 0 is the identity for $*$.

(iii) Let $a \in R$ If possible, let $x$ be the inverse of $a$. Then, by definition.

$$a * x = 0 \text{ (identity)}$$

$\Rightarrow \quad a + x - ax = 0 \quad \Rightarrow x(a - 1) = a$

$\Rightarrow \quad x = \dfrac{a}{a+1} \in R$ If $a \neq 1$

$\therefore$ $a \in R\ (a \neq 1)$ is invertible.

■ **Example 36. On the set $Q^+$ of positive rational, two binary operations are defined as follows.**

**(i) $a*b = \dfrac{ab}{3}$, $\forall\ a,b \in Q^+$ (ii) $a*b = \dfrac{ab}{4}$, $\forall\ a,b \in Q^+$**

**Prove that both the operations are commutative and associative. Find their identities and find the inverse of element with respect to these operations.**

**Sol.** $\quad a * b = \dfrac{ab}{3}, \forall\ a, b \in Q^+$

**Commutative :** $\quad a * b = \dfrac{ab}{3} = \dfrac{ba}{3} = b * a$

**Associative :** $\quad a * (b * c) = (a * b) * c$

$$\text{L.H.S.} = a * (b * c) = a * \left(\frac{bc}{3}\right)$$



$$\frac{abc}{9} = \left(\frac{ab}{3}\right) * c = (a * b) * c$$

Hence $*$ is commutative as well as associative.

**Identity element :** - Let $e$ be the identity element in the binary operation $*$. Then for $a \in Q^+$

$$a * e = \frac{ae}{3} = a$$

$\Rightarrow \quad \left(\dfrac{e}{3} - 1\right) a = 0 \qquad [\because a \neq 0]$

$\Rightarrow \quad \dfrac{e}{3} - 1 = 0 \quad \Rightarrow \quad e = 3$

Let $a \in Q^+$ If possible let $x$ be inverse element of a then from definition

$$a * x = 3 \text{ (identity element)}$$

$$\frac{ax}{3} = 3 \quad \Rightarrow \quad ax = 9 \quad \Rightarrow \quad x = \frac{9}{a}$$

**(ii)** $\quad a * b = \dfrac{ab}{4} \ \forall\ a, b \in Q^+$

**Commutative :** $\quad a * b = \dfrac{ab}{4}$

$$= \frac{ba}{4} = b * a$$

**Associative :** $\quad a * (b * c) = (a * b) * c$

$$\text{L.H.S.} = a * (b * c)$$

$$a * \left(\frac{bc}{4}\right) = \frac{abc}{16}$$

R.H.S. $\rightarrow (a * b) * c$

$$\left(\frac{ab}{4}\right) * c = \frac{abc}{16}$$

Hence $*$ is commutative as well associative operation.

**Identity element :-** Let e be the identity element in the binary operation. $*$. then for $a \in Q^+$

$$a * e = \frac{ae}{4} = a \text{ (by definition)}$$

$\Rightarrow \quad \left(\dfrac{ae}{4} - a\right) = 0$

$$\Rightarrow \quad \left(\frac{e}{4}-1\right)a=0 \qquad [\because a \neq 0]$$

$$\Rightarrow \quad \frac{e}{4}-1=0 \quad \Rightarrow \quad e=4$$

Let $a \in Q^+$ if possible let $x$ be inverse element of a then from definition.

$$a * x = 4$$

$$\frac{ax}{4} = 4 \quad \Rightarrow \quad x = \frac{16}{a} \text{ Ans.}$$

■ **Example 37. If $S = \{f_1, f_2, f_3, f_4\}$ whose $f_1, f_2, f_3$ and $f_4$ are functions defined on the set $R_0$ of non zero real numbers, as follows.**

$$f_1(x) = x,\ f_2(x) = -x,\ f_3(x) = -\frac{1}{x} \text{ and } f_4(x) = \frac{1}{x}$$

**Prepare the composition table of S for the composite of function as binary operation with the help of the composition table. Find the identity of the operation. Also find which elements are invertible. Also find the inverse.**

**Sol.** Here we have $f_1(x) = x,\ f_2(x) = -x,\ f_3(x) = \frac{-1}{x}$ and $f_4(x) = \frac{1}{x}$.

$$f_1 0 f_1(x) = f_1[f_1(x)] = f_1(x) = x$$

$\therefore$

$$f_1 0 f_2(x) = f_1[f_2(x)] = f_1(-x) = -x$$

$$f_1 0 f_3(x) = f_1[f_3(x)] = f_1\left(\frac{-1}{x}\right) = -\frac{1}{x}$$

$$f_1 0 f_4(x) = f_1[f_4(x)] = f_1\left[-\frac{1}{x}\right] = -\frac{1}{x}$$

$$f_2 0 f_1(x) = f_2[f_1(x)] = f_2(x) = -x$$

$$f_2 0 f_2(x) = f_2[f_2(x)] = f_2(-x) = x$$

$$f_2 0 f_3(x) = f_2[f_3(x)] = f_2\left(-\frac{1}{x}\right) = \frac{1}{x}$$

$$f_2 0 f_4(x) = f_2[f_4(x)] = f_2\left(\frac{1}{x}\right) = -\frac{1}{x}$$

$$f_3 0 f_1(x) = f_3[f_1(x)] = f_3(x) = -\frac{1}{x}$$

$$f_3 0 f_2(x) = f_3[f_2(x)] = f_3(-x) = \frac{1}{x}$$

$$f_3 0 f_3(x) = f_3[f_2(x)] = f_3\left(-\frac{1}{x}\right) = x$$

$$f_3 0 f_4(x) = f_3[f_4(x)] = f_3\left(\frac{1}{x}\right) = -x$$

$$f_4 0 f_1(x) = f_4[f_1(x)] = f_4(x) = \frac{1}{x}$$



$$f_4 0 f_2(x) = f_4[f_2(x)] = f_4(-x) = -\frac{1}{x}$$

$$f_4 0 f_3(x) = f_4[f_3(x)] = f_4\left(-\frac{1}{x}\right) = -x$$

$$f_4 0 f_4(x) = f_4[f_4(x)] = f_4\left(\frac{1}{x}\right) = x$$

$\therefore$ Composition table of S for the composite functions as binary operations.

| 0 | $f_1$ | $f_2$ | $f_3$ | $f_4$ |
|---|---|---|---|---|
| $f_1$ | $f_1$ | $f_2$ | $f_3$ | $f_4$ |
| $f_2$ | $f_2$ | $f_1$ | $f_4$ | $f_3$ |
| $f_3$ | $f_3$ | $f_4$ | $f_1$ | $f_2$ |
| $f_4$ | $f_4$ | $f_3$ | $f_2$ | $f_1$ |

From the above table it is clear that $x$ is the identity element of $f_1, f_2, f_3$ and $f_4$ and $f_1^{-1}(x) = x$

$$f_2^{-1}(x) = -x,\ f_3^{-1}(x) = \frac{-1}{x},\ f_4^{-1}(x) = \frac{1}{x} \qquad \text{Ans.}$$

■ **Example 38. A = {±2, ±1, 0}; B = {0, 1, 2, 3, 4}**
**f : A → B s.t. $f(x) = x^2$**
**Find type or nature of function.** **[R.U. 2016]**

**Solution :** f = {(–2, 4) (–1, 1), (0, 0), (1, 1), (2, 4)}
It is a many one and onto function.

■ **Example 39. $f(x) = 4x^2 + 3x - 5$ and $g(x) = 2x + 6$ find fog (4) and gof (–2)** **[R.U. 2016]**

**Solution :**

$$\begin{aligned} fog(4) &= f[g(4)] = f[8+6] = f(14) \\ &= 4(14)^2 + 3(14) - 5 \\ &= 821 \\ gof(-2) &= g[f(-2)] \\ &= g[4 \times 4 + 3 \times -2 - 5] \\ &= g(16 - 6 - 5) = g(5) = 2 \times 5 + 6 \\ &= 16 \end{aligned}$$

## EXERCISE 2.1

1. Let A and B sets. Show that $f : A \times B \to B \times A$ such that f(a, b) is bijective functions.
2. If $f(x) = \frac{4x+3}{6x-4}$; $x \neq \frac{2}{3}$ show that $fof = x\ \forall\ x \neq \frac{2}{3}$.
3. Consider $f : R_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x$ show that $f$ is invertible with

$$f(y) = \frac{\sqrt{y+6}-1}{3}.$$

4. If $f : R \to R$ is defined by $f(x) = x^2 - 3x + 2$; find $f(f(x))$.

5. Let $x$ be a binary operation on the set Q of rotational number as $a * b = \frac{ab}{4}$. Check whether * it commutative and associative?

6. Show that $a * b = \frac{a+b}{2}$ $\forall a, b \in R$ is commutative but not associative.

7. Let $f : X \to Y$ be an invertible function. Show that the inverse of $f^{-1}$ is $f$, i.e. $(f^{-1})^{-1} = f$.

■■■